COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2021.

Lecture 2

Reminders:

- Sign up for Piazza – there has already been a lot of great discussion.
- Find homework teammates and sign up for Gradescope (code on course website).
- My office hours (on Zoom) have moved to Thursday, 9:00am-10:30am.

Last Class We Covered:

- Basic probability review. See course site for links to resources to refresh your probability background.
- Linearity of expectation: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ always.
- Linearity of variance: $\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ if X and Y are independent.

Today:

- An algorithmic application of of linearity of expectation and variance.
- Introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

Let $X = X_1 + X_2 + X_3$ where X_1, X_2, X_3 are independent random variables, each with expectation 5 and variance 1.

What is $\mathbb{E}(X/3)$?

$$\mathbb{E}\left(\frac{X_1 + X_2 + X_3}{3}\right) = 5$$

$$E(X_1 + X_2 + X_3) = \mathbb{E}X_1 + \mathbb{E}X_2 + \mathbb{E}X_3 = 15$$

$$Var(\alpha \cdot X) = \alpha^2 \cdot Var(X)$$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$Var(\alpha X) = \alpha^2 \mathbb{E}X^2 - \alpha^2 (\mathbb{E}X)^2 = \alpha^2 Var(X)$$

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What is $Var(X/3)$?

$$Var\left(\frac{X_1 + X_2 + X_3}{3}\right) = \frac{1}{3}$$

$$\frac{1}{9} Var(X_1 + X_2 + X_3) = \frac{1}{9} \cdot (1+1+1) = \frac{1}{3}$$

11

The expected number of inches of rain on Saturday is 6 and the expected number of inches on Sunday is 5. There is a 50% chance of rain on Saturday. If it rains on Saturday, there is a 75% chance of rain on Sunday. If it does not rain on Saturday, there is only a 25% chance of rain on Sunday. What is the expected number of inches of rainfall total over the weekend?

$E(SAT + SUN) = E(SAT) + E(SUN)$

$6 + 5 = 11$

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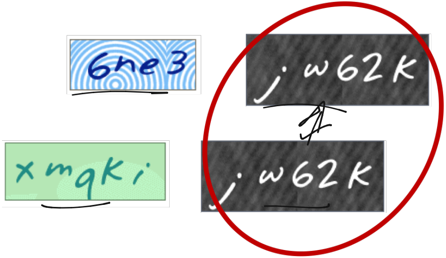
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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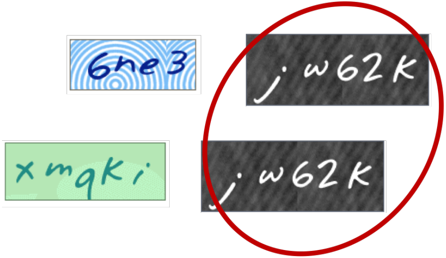

'Mark and recapture' method in ecology.

AN ALGORITHMIC APPLICATION

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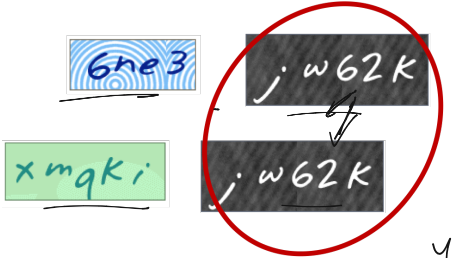

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Think-Pair-Share: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i, j), (i, k) and (j, k).

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

n: number of CAPTCHAS in database, m: number of random CAPTCHAS drawn to check database size, $\mathbf{D}$: number of pairwise duplicates in m random CAPTCHAS

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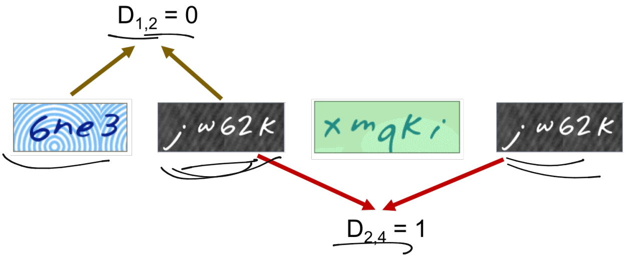

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$$D = \sum_{\substack{i,j \in [m], i \neq j \\ i < j}} D_{i,j}.$$

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$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[\mathbf{D}_{i,j}].$$

$i < j$

$\mathbb{E}[D_{ij}] = \frac{1}{n}$

$\mathbb{E}\, D_{ij} = P(D_{ij} = 1) \cdot 1 + P(D_{ij} = 0) \cdot 0$

$\frac{1}{n} \cdot 1 + 0$

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LINEARITY OF EXPECTATION

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable. The number of pairwise duplicates (a random variable) is:

$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[\mathbf{D}_{i,j}].$$

For any pair $i, j \in [m], i \neq j$: $\mathbb{E}[\mathbf{D}_{i,j}] = \Pr[\mathbf{D}_{i,j} = 1] = \frac{1}{n}$.

$$\mathbb{E}[D] = \sum_{\substack{i,j \in [n] \\ i<j}} \frac{1}{n} = \binom{m}{2} \cdot \frac{1}{n} = \frac{m(m-1)}{2} \cdot n$$

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$\frac{m}{n} \sim \frac{m^2}{n^2} \cdot n = \frac{m^2}{n}$

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$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}. \quad \sim O\left(\frac{m^2}{n}\right)$$

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Note that the $\mathbf{D}_{i,j}$ random variables are not independent!

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If there are a 110 people in this room, each whose birthday we assume to be a uniformly random day of the 365 days in the year, how many pairwise duplicate birthdays do we expect there are?

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$$\mathbb{E}[\mathsf{D}] = \frac{m(m-1)}{2n} = \frac{110 \cdot 109}{2 \cdot 365} \approx 16.5.$$

LINEARITY OF EXPECTATION

You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

1000·(1000 -1)

$$\mathbb{E}[\mathbf{D}] = \frac{m(m-1)}{2n} = .4995$$

2·1000,000

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The most fundamental concentration bound: **Markov's inequality.**

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For any non-negative random variable X and any $t > 0$:

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

$\mathbb{E}[X] = .5$

$\Pr(X \geq 10) \leq \frac{.5}{10} \leq .05$

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For any non-negative random variable X and any $t > 0$:

$$\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}.$$

Proof:

$$\mathbb{E}[X] = \sum_{s} \Pr(X = s) \cdot s \geq \sum_{s \geq t} \Pr(X = s) \cdot s$$

$$\Pr(S \geq 10) \leq \frac{7}{10}$$

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The larger the deviation t, the smaller the probability.

Expected number of duplicate CAPTCHAS:

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hash-map
dict

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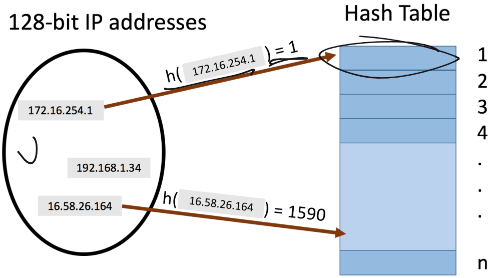
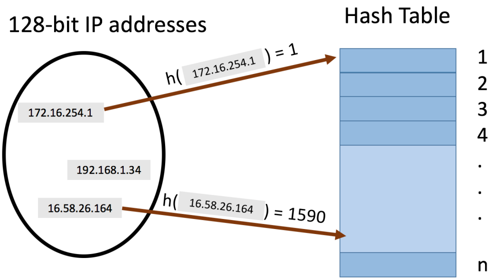
Classic Solution: Hash tables

- *Static hashing* since we won't worry about insertion and deletion today.



- **hash function** $h : U \to [n]$ maps elements from the universe to indices $1, \cdots, n$ of an array.



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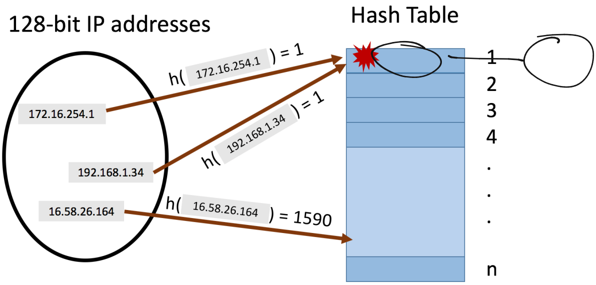

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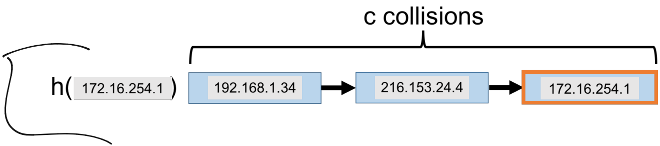
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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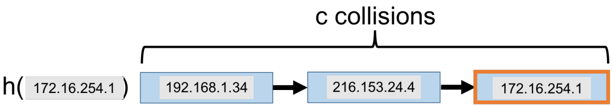

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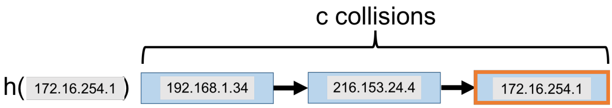

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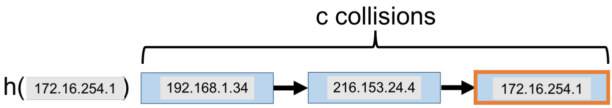

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- In the worst case could have $c = m$ (all items hash to the same location).
- Two approaches: 1) we assume the items inserted are chosen randomly from the universe U or 2) we assume the hash function is random.

RANDOM HASH FUNCTION

Let $\mathbf{h} : U \to [n]$ be a fully random hash function.

- I.e., for $x \in U$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ are independent for any two items $x \neq y$.

$h(128.00) = 2$

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

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Think-Pair-Share: Assuming we insert m elements into a hash table of size n, what is the expected total number of pairwise collisions? (using a fully random hash function.

Let $C_{i,j} = 1$ if items i and j collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$C = \sum_{i,j \in [m], i \neq j} C_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, h: random hash function.

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x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table, $\mathbf{h}$: random hash function.

Let $C_{i,j} = 1$ if items i and j collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$\mathbb{E}[C] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[C_{i,j}]. \qquad \text{(linearity of expectation)}$$

For any pair i, j, $i \neq j$:

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$$\mathbb{E}[C] = \sum_{i,j \in [m], i \neq j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

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Identical to the CAPTCHA analysis!

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storage items

- For $n = 4m^2$ we have: $\mathbb{E}[\mathrm{C}] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}$.

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- **Think-Pair-Share:** Give a lower bound on the probability that we have no collisions, i.e., $\Pr[\mathbf{C} = 0]$?

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Apply Markov's Inequality: $\Pr[C \geq 1] \leq \frac{\mathbb{E}[C]}{1} \leq \frac{1}{8}$

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Pretty good...but we are using $O(m^2)$ space to store m items...

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Questions?