COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2021.

Lecture 17

$A = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$ $\quad A^T A = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

- Problem Set 3 is due next Monday 11/8, 12:59pm.
- For Piazza participation credit, posts must be public. It is ok if they are anonymous to your classmates (none are anonymous to us).
- A number of people asked for mid-level practice questions bridging the quizzes and homeworks. I will try to post more of those. I will post some linear algebra ones in a few days.
- When tackling the homework problems, before you begin trying to prove anything, really make sure you understand the definitions (E.g., are variables scalars or matrices. If matrices, what dimension are they? If scalars, what possible range of values could they take?) To me it is always helpful to draw out examples.

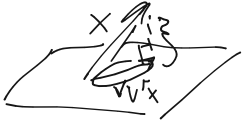
$\underline{tr(A^T A)} = \underline{\|A\|_F^2}$

SUMMARY

Last Few Classes: Low-Rank Approximation and PCA

- Compress data that lies close to a k-dimensional subspace.
- Equivalent to finding a low-rank approximation of the data matrix $\mathbf{X}$: $\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T$ for orthonormal $\mathbf{V} \in \mathbb{R}^{d \times k}$.
- Optimal solution via eigendecomposition of $\mathbf{X}^T\mathbf{X}$.
- Error analysis by looking at the eigenvalue spectrum of $\mathbf{X}^T\mathbf{X}$.
  $\min_{\mathbf{B}:\text{rank}(\mathbf{B}) \le k} \|\mathbf{X} - \mathbf{B}\|_F^2 = \sum_{i=k+1}^{d} \lambda_i(\mathbf{X}^T\mathbf{X})$.

Q3 Quiz: $\|VV^Tx\|_2^2 \le \|x\|_2^2$ ALWAYS SOMETIME NEVER



$$\|VV^Tx\|_2^2 + \underbrace{\|(I-VV^T)x\|_2^2}_{\ge 0} = \|x\|_2^2$$

$$\|VV^Tx\|_2^2 \le \|x\|_2^2$$

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- Optimal solution via eigendecomposition of $\mathbf{X}^T\mathbf{X}$.
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This Class: The SVD and Applications of low-rank approximation.

- The singular value decompostion (SVD) and its connections to eigendecomposition and low-rank approximation.
- Matrix completion and collaborative filtering
- Entity embeddings (word embeddings, node embeddings, etc.)

The Singular Value Decomposition (SVD) generalizes the eigendecomposition to asymmetric (even rectangular) matrices.

$n \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} \to \begin{bmatrix} Ax \end{bmatrix}$

Q5 Quiz:

$Av = 0 \cdot v = 0$

$\begin{bmatrix} Ax \end{bmatrix} = \begin{bmatrix} \lambda x \\ 0 \end{bmatrix}$

What is the maximum eigenvalue of

$2I - A$

$\lambda_1(A) = 5, \quad \lambda_2(A) = 2 \quad \lambda_3(A) = -1$

same eigenvectors as A

$\lambda_1(2I - A) = 3, \quad \lambda_2(2I - A) = 0,$

$\lambda_3(2I - A) = 3$

let v be an eigenvector of A eigenvalue λ

$(2I - A)v = 2v - Av = 2v - \lambda v$

$= (2 - \lambda)v$

The Singular Value Decomposition (SVD) generalizes the eigendecomposition to asymmetric (even rectangular) matrices. Any matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ with $\text{rank}(\mathbf{X}) = r$ can be written as $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$.

$r \leq \min(n, d)$

- $\mathbf{U}$ has orthonormal columns $\vec{u}_1, \ldots, \vec{u}_r \in \mathbb{R}^n$ (left singular vectors).
- $\mathbf{V}$ has orthonormal columns $\vec{v}_1, \ldots, \vec{v}_r \in \mathbb{R}^d$ (right singular vectors).
- $\boldsymbol{\Sigma}$ is diagonal with elements $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ (singular values).

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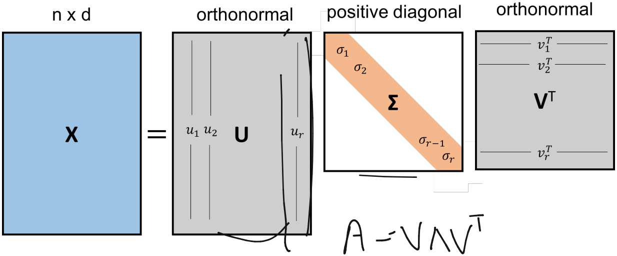

$A = V \Lambda V^T$

The Singular Value Decomposition (SVD) generalizes the eigendecomposition to asymmetric (even rectangular) matrices. Any matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ with $\text{rank}(\mathbf{X}) = r$ can be written as $\mathbf{X} = \mathbf{U\Sigma V}^T$.

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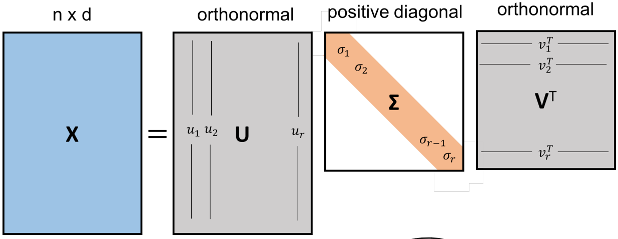

The ‘swiss army knife’ of modern linear algebra.

CONNECTION OF THE SVD TO EIGENDECOMPOSITION

Writing $\mathbf{X} \in \mathbb{R}^{n \times d}$ in its singular value decomposition $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$:

$$\mathbf{X}^T\mathbf{X} =$$

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

Writing $X \in \mathbb{R}^{n\times d}$ in its singular value decomposition $X = U\Sigma V^T$:

$$X^TX = \overset{d\times n}{\underbrace{V\Sigma U^T}_{X^T}}\;\overset{n\times d}{\underbrace{U\Sigma V^T}_{X}} = V\Sigma^2V^T$$

I

$X \in \mathbb{R}^{n\times d}$: data matrix, $U \in \mathbb{R}^{n\times \text{rank}(X)}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $V \in \mathbb{R}^{d\times \text{rank}(X)}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\boldsymbol{\Sigma} \in \mathbb{R}^{\text{rank}(X)\times \text{rank}(X)}$: positive diagonal matrix containing singular values of X.

CONNECTION OF THE SVD TO EIGENDECOMPOSITION

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$$\mathbf{X}^T\mathbf{X} = \mathbf{V}\boldsymbol{\Sigma}\underbrace{\mathbf{U}^T\mathbf{U}}_{I}\boldsymbol{\Sigma}\mathbf{V}^T = \mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^T$$

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\boldsymbol{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

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$$\mathbf{X}^T\mathbf{X} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^T\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T = \mathbf{V}\boldsymbol{\Sigma}^2\mathbf{V}^T \text{ (the eigendecomposition)}$$

orthonormal

diagonal

columns of V, are eigenvectors of X^TX $\sigma_i^2(X)$ are eigenvalues

Exercise: Prove that since $X^TX = V\Sigma^2V^T$ that V_i is an eigenvector of X^TX with eigenvalue $\sigma_i(X)^2$

$X^TXV_i = V\Sigma^2V^TV_i = \sigma_i(X)^2 \cdot V_i$

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\boldsymbol{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

Writing $\mathbf{X} \in \mathbb{R}^{n \times d}$ in its singular value decomposition $\mathbf{X} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$:

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Similarly: $\underbrace{\mathbf{X}\mathbf{X}^T} = \underbrace{\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T}_{\mathbf{X}}\underbrace{\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^T}_{\mathbf{X}^T} = \underbrace{\mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^T}$.

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\boldsymbol{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

$\Sigma \in \mathbb{R}^{r\times r}$

Writing $\mathbf{X} \in \mathbb{R}^{n\times d}$ in its singular value decomposition $\mathbf{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$:

$d\times r \quad r\times r \quad r\times d$

$$\mathbf{X}^T\mathbf{X} = \mathbf{V}\mathbf{\Sigma}\mathbf{U}^T\mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \mathbf{V}\mathbf{\Sigma}^2\mathbf{V}^T \text{ (the eigendecomposition)}.$$

$r\times r$

$n\times r \quad r\times n$

Similarly: $\mathbf{X}\mathbf{X}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}\mathbf{U}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T$.

orthonormal

$[U][\Sigma^2][U^T]$

The left and right singular vectors are the eigenvectors of the ~~covariance matrix~~ $\mathbf{X}^T\mathbf{X}$ and the ~~gram matrix~~ $\mathbf{X}\mathbf{X}^T$ respectively.

$\mathbf{X} \in \mathbb{R}^{n\times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n\times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d\times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X})\times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

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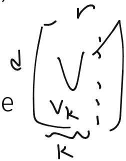

The left and right singular vectors are the eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$ and the gram matrix $\mathbf{X}\mathbf{X}^T$ respectively.

So, letting $\mathbf{V}_k \in \mathbb{R}^{d \times k}$ have columns equal to $\vec{v}_1, \ldots, \vec{v}_k$, we know that $\mathbf{X}\mathbf{V}_k\mathbf{V}_k^T$ is the best rank-k approximation to $\mathbf{X}$ (given by PCA).

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

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Similarly: $\mathbf{X}\mathbf{X}^T = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T\mathbf{V}\boldsymbol{\Sigma}\mathbf{U}^T = \mathbf{U}\boldsymbol{\Sigma}^2\mathbf{U}^T$.

The left and right singular vectors are the eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$ and the gram matrix $\mathbf{X}\mathbf{X}^T$ respectively.

So, letting $\mathbf{V}_k \in \mathbb{R}^{d \times k}$ have columns equal to $\vec{v}_1, \ldots, \vec{v}_k$, we know that
$n \times d$ $d \times d$
$\mathbf{X}\mathbf{V}_k\mathbf{V}_k^T$ is the best rank-k approximation to $\mathbf{X}$ (given by PCA).
$n \times n$ $n \times d$

What about $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X}$ where $\mathbf{U}_k \in \mathbb{R}^{n \times k}$ has columns equal to $\vec{u}_1, \ldots, \vec{u}_k$?

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\boldsymbol{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

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Similarly: $\mathbf{X}\mathbf{X}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}\mathbf{U}^T = \mathbf{U}\mathbf{\Sigma}^2\mathbf{U}^T$.

The left and right singular vectors are the eigenvectors of the covariance matrix $\mathbf{X}^T\mathbf{X}$ and the gram matrix $\mathbf{X}\mathbf{X}^T$ respectively.

So, letting $\mathbf{V}_k \in \mathbb{R}^{d \times k}$ have columns equal to $\vec{v}_1, \ldots, \vec{v}_k$, we know that $\mathbf{X}\mathbf{V}_k\mathbf{V}_k^T$ is the best rank-k approximation to $\mathbf{X}$ (given by PCA). $XV_kV_k^T = U_kU_k^TX$

What about $\mathbf{U}_k\mathbf{U}_k^T\mathbf{X}$ where $\mathbf{U}_k \in \mathbb{R}^{n \times k}$ has columns equal to $\vec{u}_1, \ldots, \vec{u}_k$?
Gives exactly the same approximation!

> $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

The best low-rank approximation to $\mathbf{X}$:
$\mathbf{X}_k = \arg\min_{\text{rank}-k\ \mathbf{B}\in\mathbb{R}^{n\times d}} \|\mathbf{X}-\mathbf{B}\|_F$ is given by:

$$\mathbf{X}_k = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T = \mathbf{U}_k\mathbf{U}_k^T\mathbf{X}$$

$\overset{d\times d}{V_k V_k^T}$ → projects rows to the span top k right singular vectors

$\overset{n\times n}{U_k U_k^T}$ ← projects columns " left singular vctrs

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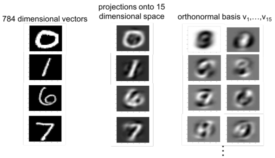
Correspond to projecting the rows (data points) onto the span of $\mathbf{V}_k$ or the columns (features) onto the span of $\mathbf{U}_k$

Row (data point) compression



Column (feature) compression

10000* bathrooms+ 10* (sq. ft.) ≈ list price

	bedrooms	bathrooms	sq.ft.	floors	list price	sale price
home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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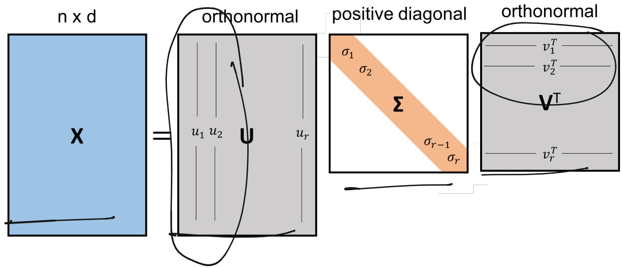

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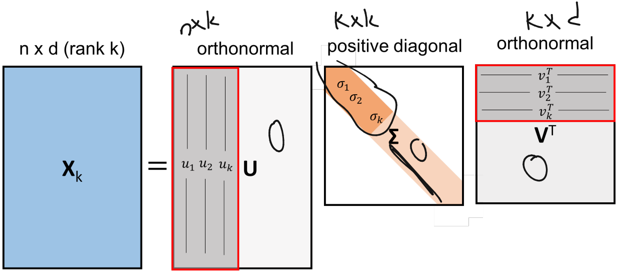

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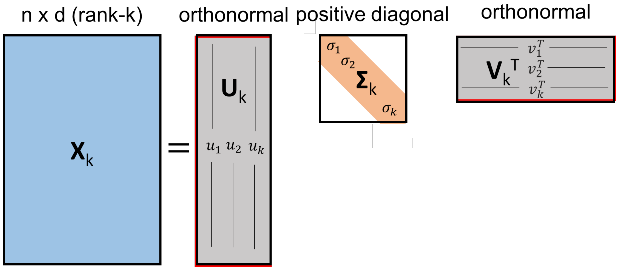

THE SVD AND OPTIMAL LOW-RANK APPROXIMATION

any A, B
eigenvalues $A \cdot B$ are same as $B \cdot A$

The best low-rank approximation to $\mathbf{X}$:
$\mathbf{X}_k = \arg\min_{\text{rank}-k\ \mathbf{B} \in \mathbb{R}^{n \times d}} \|\mathbf{X} - \mathbf{B}\|_F$ is given by:

$$\underbrace{\mathbf{X}_k = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T}_{\text{know}} = \mathbf{U}_k\mathbf{U}_k^T\mathbf{X} = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^T$$

(I'll show this)

$\underbrace{X^T}_{A}\underbrace{X}_{B}$ $\underbrace{X}_{B}\underbrace{X^T}_{A}$ have same nonzero eigenvalues.

$\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d \times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X}) \times \text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

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$$\mathbf{X}_k = \mathbf{X}\mathbf{V}_k\mathbf{V}_k^T = \mathbf{U}_k\mathbf{U}_k^T\mathbf{X} = \mathbf{U}_k\mathbf{\Sigma}_k\mathbf{V}_k^T$$

$$XV_kV_k^T = \underset{n\times r}{U}\,\underset{r\times r}{\Sigma}\,\underset{r\times d}{V^T}\,\underset{d\times k}{V_k}\,\underset{k\times d}{V_k^T}$$

$$= U\Sigma_r \begin{bmatrix} I \\ 0 \end{bmatrix} V_k^T = U_k\Sigma_kV_k^T$$

$$n\,[U\Sigma]_k = \begin{bmatrix} | & | & & | \\ u_1\sigma_1 & u_2\sigma_2 & \cdots & u_k\sigma_k \\ | & | & & | \end{bmatrix} = U_k\Sigma_k = \begin{bmatrix} U_k \\ u_1\,u_2\cdots u_k \end{bmatrix}\begin{bmatrix} \sigma_1 & & \Sigma_k \\ & \sigma_2 & \\ & & \sigma_k \end{bmatrix}$$

$$\underset{r\times k}{\begin{bmatrix} I_{k\times k} \\ 0 \end{bmatrix}}\begin{bmatrix} V_k^T \end{bmatrix}_k = \begin{bmatrix} V_k^T \\ 0 \end{bmatrix}$$

$\mathbf{X} \in \mathbb{R}^{n\times d}$: data matrix, $\mathbf{U} \in \mathbb{R}^{n\times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{u}_1, \vec{u}_2, \ldots$ (left singular vectors), $\mathbf{V} \in \mathbb{R}^{d\times \text{rank}(\mathbf{X})}$: matrix with orthonormal columns $\vec{v}_1, \vec{v}_2, \ldots$ (right singular vectors), $\mathbf{\Sigma} \in \mathbb{R}^{\text{rank}(\mathbf{X})\times\text{rank}(\mathbf{X})}$: positive diagonal matrix containing singular values of $\mathbf{X}$.

Applications of low-rank approximation beyond compression.

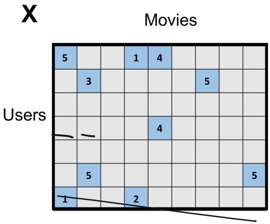
Consider a matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ which we cannot fully observe but believe is close to rank-k (i.e., well approximated by a rank k matrix).

Consider a matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ which we cannot fully observe but believe is close to rank-k (i.e., well approximated by a rank k matrix). Classic example: the Netflix prize problem.

$$\mathbf{X} \qquad \begin{array}{cc} & \text{Movies} \\ \text{Users} & \begin{bmatrix} 5 & 3 & 3 & 1 & 4 & 4 & 4 & 3 & 5 \\ 4 & 3 & 3 & 1 & 4 & 4 & 5 & 3 & 5 \\ 3 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 \\ 4 & 3 & 3 & 4 & 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 3 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 5 & 3 & 4 & 4 & 4 & 4 & 4 & 5 \\ 1 & 3 & 3 & 2 & 3 & 3 & 3 & 1 & 2 \end{bmatrix} \end{array}$$

MATRIX COMPLETION

Consider a matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ which we cannot fully observe but believe is close to rank-k (i.e., well approximated by a rank k matrix). Classic example: the Netflix prize problem.



Consider a matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ which we cannot fully observe but believe is close to rank-k (i.e., well approximated by a rank k matrix). Classic example: the Netflix prize problem.

X Movies

Assume rank(**X**)=1

Users

5	2	1	1	4
10	4	2	2	8
10	4	2	2	8
5	2	1	1	4

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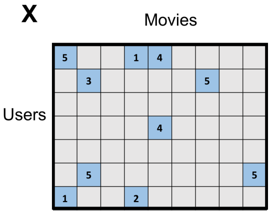

Solve: $Y = \underset{\text{rank}-k\ \mathbf{B}}{\arg\min} \sum_{\text{observed } (j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$

$\|X - B\|_F^2$

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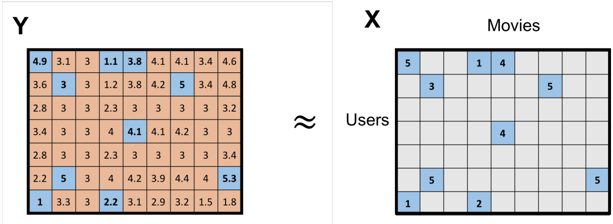

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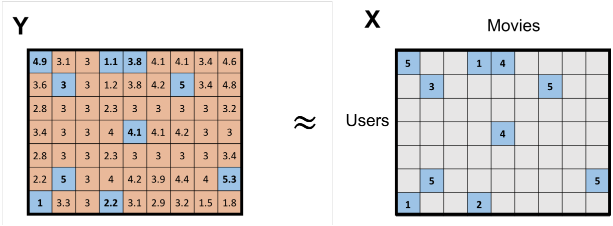

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Under certain assumptions, can show that $\mathbf{Y}$ well approximates $\mathbf{X}$ on both the observed and (most importantly) unobserved entries.

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Y

4.9	3.1	3	**1.1**	**3.8**	4.1	4.1	3.4	4.6
3.6	**3**	3	1.2	3.8	4.2	**5**	3.4	4.8
2.8	3	3	2.3	3	3	3	3	3.2
3.4	3	3	4	**4.1**	4.1	4.2	3	3
2.8	3	3	2.3	3	3	3	3	3.4
2.2	**5**	3	4	4.2	3.9	4.4	4	**5.3**
1	3.3	3	**2.2**	3.1	2.9	3.2	1.5	1.8

$\approx$

X Movies

Users

5	3	3	1	4	4	4	3	5
4	3	3	1	4	4	5	3	5
3	3	3	2	3	3	3	3	3
4	3	3	4	4	4	4	3	3
3	3	3	2	3	3	3	3	3
2	5	3	4	4	4	4	4	5
1	3	3	2	3	3	3	1	2

Solve: $Y = \underset{\text{rank}-k\ \mathbf{B}}{\arg\min} \sum_{\text{observed } (j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$

Under certain assumptions, can show that $\mathbf{Y}$ well approximates $\mathbf{X}$ on both the observed and (most importantly) unobserved entries.