COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

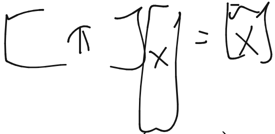
University of Massachusetts Amherst. Fall 2021.

Lecture 14

- We will be grading midterms soon, and plan to return before the add/drop deadline.
- No quiz this week.
- No office hours today.

SUMMARY

Last Few Classes:



The Johnson-Lindenstrauss Lemma

- Reduce n data points in any dimension d to $O\left(\frac{\log n/\delta}{\epsilon^2}\right)$ dimensions and preserve (with probability $\geq 1-\delta$) all pairwise distances up to $1 \pm \epsilon$.
- Compression is linear via multiplication with a random, data oblivious, matrix (linear compression)

High-Dimensional Geometry

- Why high-dimensional space is so different than low-dimensional space.
- How the JL Lemma can still work.

SUMMARY

Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).

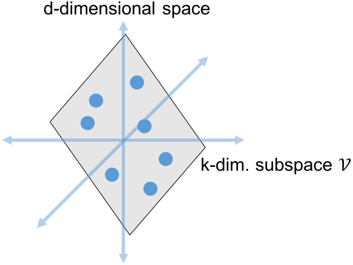
- Reduce d-dimesional data points to a smaller dimension m.
- Like JL, compression is linear – by applying a matrix.
- Chose this matrix carefully, taking into account structure of the dataset.
- Can give better compression than random projection.

Will be using a fair amount of linear algebra: orthogonal basis, column/row span, eigenvectors, etc.

RANDOMIZED ALGORITHMS UNIT TAKEAWAYS

- Randomization is an important tool in working with large datasets.
- Lets us solve 'easy' problems that get really difficult on massive datasets. Fast/space efficient look up (hash tables and bloom filters), distinct items counting, frequent items counting, near neighbor search (LSH), etc.
- The analysis of randomized algorithms leads to complex output distributions, which we can't compute exactly.
- We've covered many of the key ideas used through a small number of example applications/algorithms.
- We use concentration inequalities to bound these distributions and behaviors like accuracy, space usage, and runtime.
- Concentration inequalities and probability tools used in randomized algorithms are also fundamental in statistics, machine learning theory, probabilistic modeling of complex systems, etc.

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



EMBEDDING WITH ASSUMPTIONS

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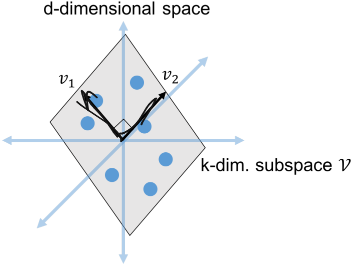

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

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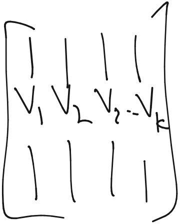

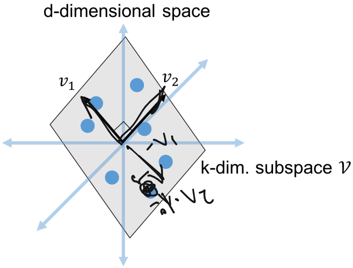

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$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

- $\mathbf{V}^T \in \mathbb{R}^{k \times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into k dimensions with no distortion.

$V = [v_1 v_2 \ldots v_k]$ (d × k)

$\begin{bmatrix} v_1^T \\ v_2^T \end{bmatrix} [v_1 v_2] = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 = 0 \\ v_2^T v_1 & v_2^T v_2 \end{bmatrix}$ (0, 1)

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j \in \mathcal{V}$:

(A) $$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

$(V(c_i - c_j))^T = (c_i - c_j)^T V^T$

for any x_i, $x_i = Vc_i$ for some $c_i \in \mathbb{R}^k$

$x_i = v_1 \cdot c_i(1) + v_2 \cdot c_i(2) + \ldots v_k \cdot c_i(k)$

$\|V^T V c_i - V^T V c_j\|_2 = \|Vc_i - Vc_j\|_2$ ($V^T V = I$)

$V^T V = I$

$(V^T V)_{ij} = v_i^T v_j$ $\quad i \neq j, = 0$

$i = j \quad = 1 = \|v_i\|_2^2$

want (B) $\|c_i - c_j\|_2 = \|V(c_i - c_j)\|_2$

(for any y, $\|y\|_2^2 = y^T y$)

$y = V(c_i - c_j)$

$\|V(c_i - c_j)\|_2^2 = (c_i - c_j)^T V^T V (c_i - c_j)$ ($V^T V = I$)

$= (c_i - c_j)^T (c_i - c_j)$

$= \|c_i - c_j\|_2^2$

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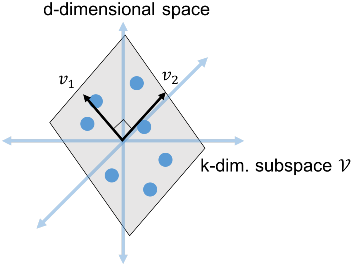

$$\|V^T x_i - V^T x_j\|_2^2 = \|x_i - x_j\|_2$$

$\downarrow$

$V^T(x_i - x_j)$

$(x_i - x_j)^T V V^T (x_i - x_j)$

$V^T V$

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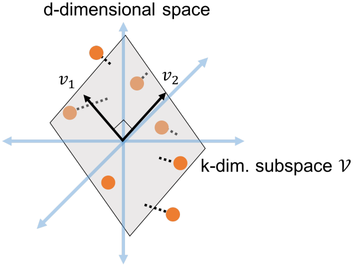

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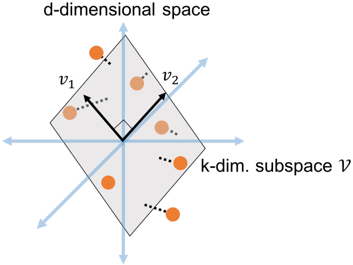

Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns, $\mathbf{V}^T\vec{x}_i \in \mathbb{R}^k$ is still a good embedding for $x_i \in \mathbb{R}^d$.

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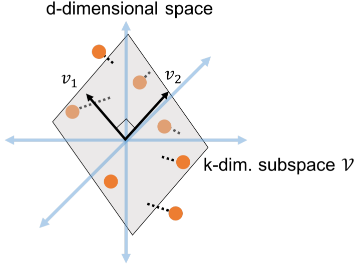

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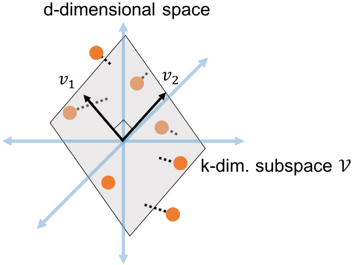

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- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

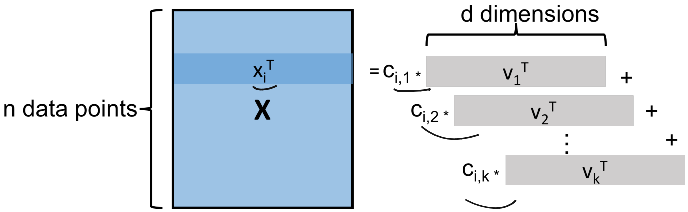
Claim: $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Claim: $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

- Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$, can write any $\vec{x}_i$ as:

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + c_{i,2} \cdot \vec{v}_2 + \ldots + c_{i,k} \cdot \vec{v}_k.$$



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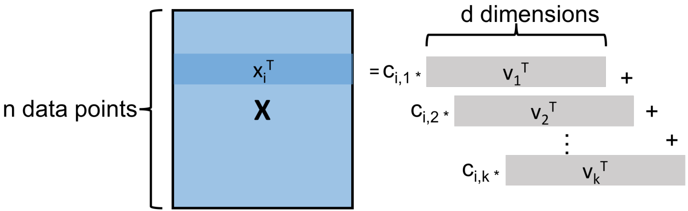
- So $\vec{v}_1, \ldots, \vec{v}_k$ span the rows of $\mathbf{X}$ and thus $\text{rank}(\mathbf{X}) \leq k$.



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- Every data point $\vec{x}_i$ (row of $\mathbf{X}$) can be written as $\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + \ldots + c_{i,k} \cdot \vec{v}_k$.

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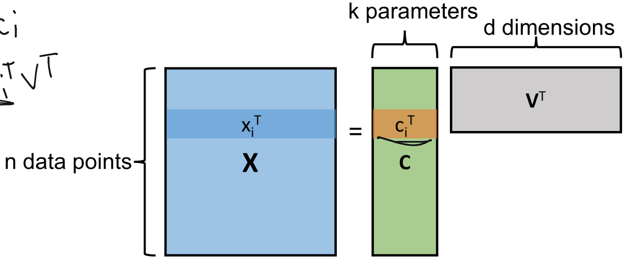
$$x_i = V c_i$$

$$x_i^T = c_i^T V^T$$



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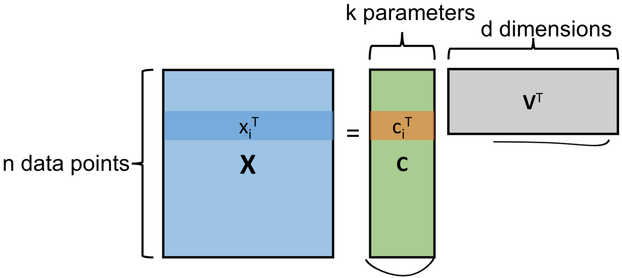

- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.

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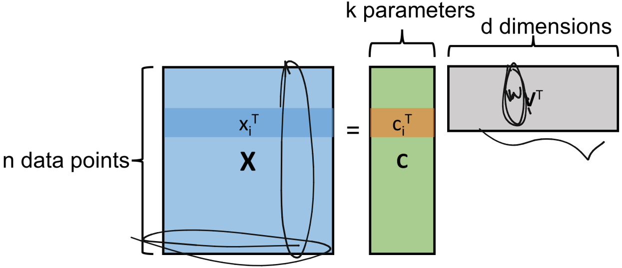

- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{V} \implies$ the columns of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{C}$.

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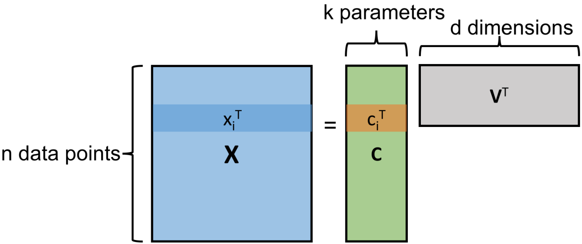
Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be written as $\mathbf{X} = \mathbf{C}\mathbf{V}^T$.



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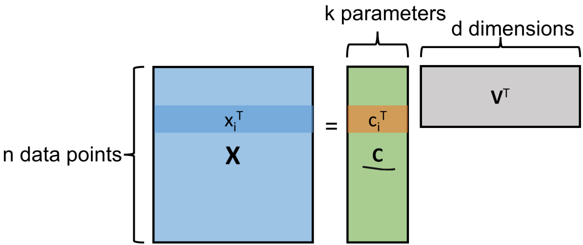

Exercise: What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

$C = XV$

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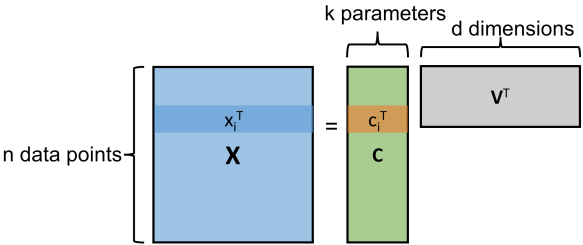

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LOW-RANK FACTORIZATION

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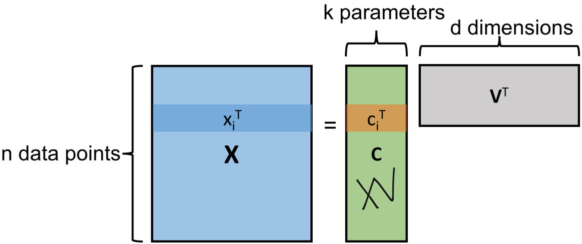

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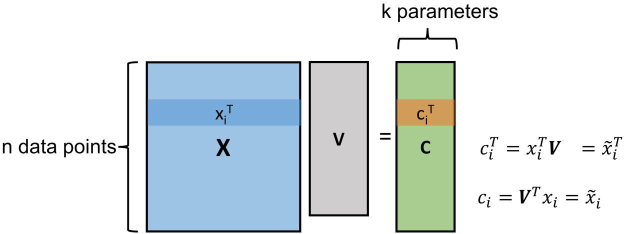

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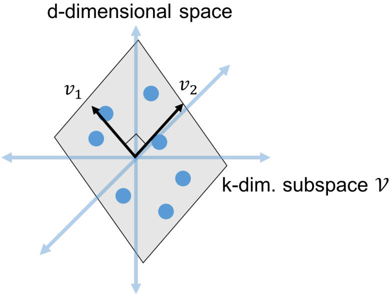

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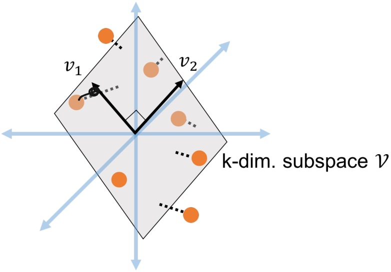

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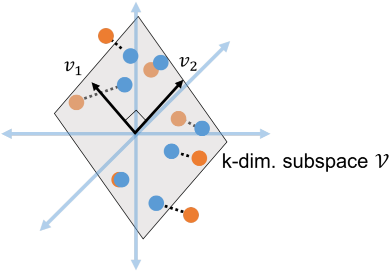

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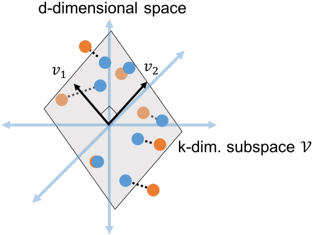

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n×d

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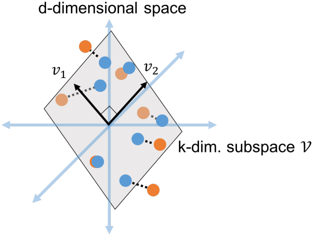

Note: $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank k. It is a low-rank approximation of $\mathbf{X}$.

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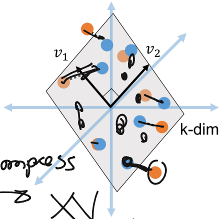

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$$\mathbf{X}\mathbf{V}\mathbf{V}^T = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F^2 = \sum_{i,j} (\mathbf{X}_{i,j} - \mathbf{B}_{i,j})^2.$$

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Key question is how to find the subspace $\mathcal{V}$ and correspondingly $\mathbf{V}$.

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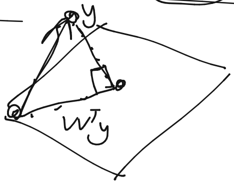
Quick Exercise: Show that $\mathbf{V}\mathbf{V}^T$ is idempotent. I.e., $(\mathbf{V}\mathbf{V}^T)(\mathbf{V}\mathbf{V}^T)\vec{y} = (\mathbf{V}\mathbf{V}^T)\vec{y}$ for any $\vec{y} \in \mathbb{R}^d$.

Why does this make sense intuitively?

Less Quick Exercise: (Pythagorean Theorem) Show that:

$$\|\vec{y}\|_2^2 = \|(\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2 + \|\vec{y} - (\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2.$$



Question: Why might we expect $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ to lie close to a k-dimensional subspace?

A STEP BACK: WHY LOW-RANK APPROXIMATION?

Question: Why might we expect $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ to lie close to a k-dimensional subspace?

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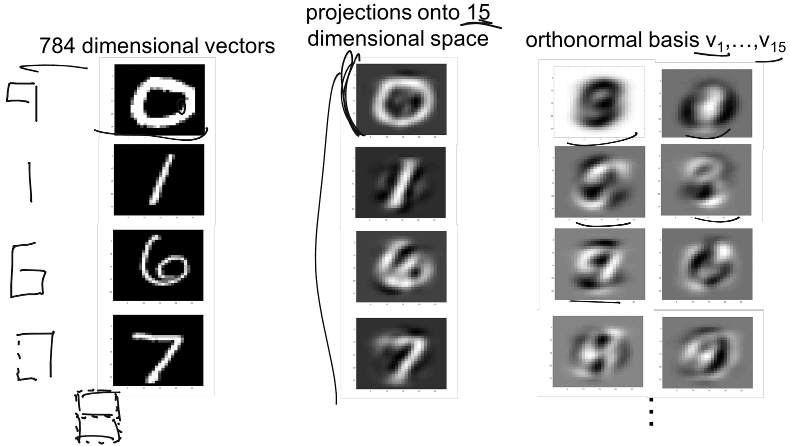

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home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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