COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2020.

Lecture 5

LOGISTICS

- Problem Set 1 is due this Friday, 9/11 at 8pm in Gradescope.
- If you can, we encourage you to make your questions public on Piazza.

Say requests were independent!

X = # failures $\Pr(X > 0) = 1 - .99^{20}$

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$1 - .99^{20}$ – we don't have independence

Quiz 2:

- Class Pace: 48% just right, 42% a bit too fast, 5% a bit too slow, 5% way too fast.

$A_1, A_2 \ldots A_{20}$

- I receive 20 download requests per day and serve each in within 15 seconds with probability 99%. Upper bound the probability I *fail to serve at least one request.*

$P(A_i) = .01$ $\Pr(A_1 \cup A_2 \cup \ldots A_{20}) \leq \sum P(A_i) = 20 \cdot .01 = .02$

Last Class: Concentration bounds beyond Markov's inequality

- Chebyshev's inequality and the law of large numbers.
- Exponential concentration bounds from higher moments.
- Bernstein's Inequality

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This Time:

- Finish up exponential concentration bounds and the central limit theorem.

- Start on algorithms: Bloom Filters

Bernstein Inequality (Simplified): Consider independent random variables $X_1, \ldots, X_n$ falling in [-1,1]. Let $\mu = \mathbb{E}[\sum X_i]$, $\sigma^2 = \text{Var}[\sum X_i]$, and $s \leq \sigma$. Then:

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

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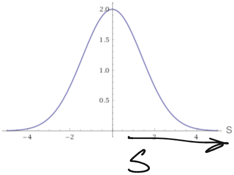

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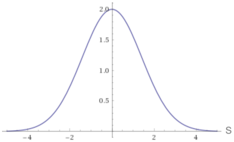

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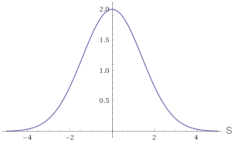

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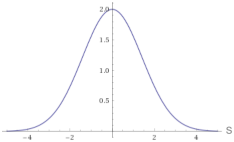

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GAUSSIAN TAILS

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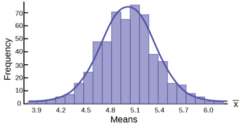
Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



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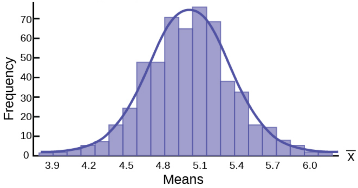

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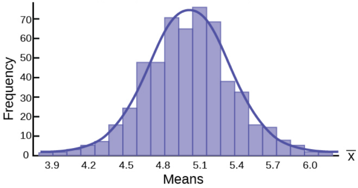

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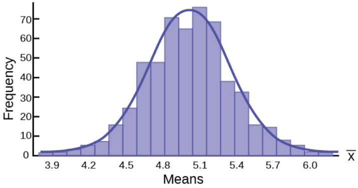

- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

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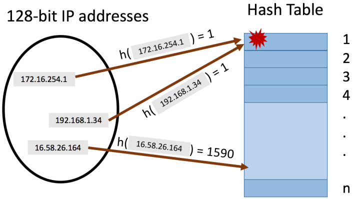
As δ gets larger and larger, the bound falls of exponentially fast.

$n = m^2$
collision free
2 level hashing
$n \sim m$



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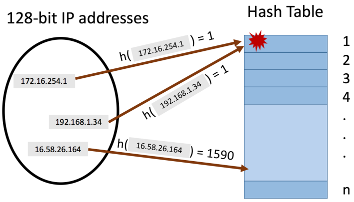

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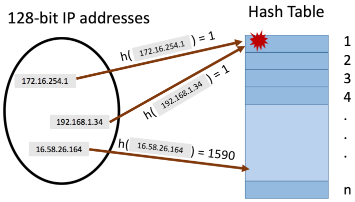

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What will be the maximum number of items hashed into the same location?

Let $\mathbf{S}_i$ be the number of items hashed into position i and $\mathbf{S}_{i,j}$ be 1 if x_j is hashed into bucket i ($\mathbf{h}(x_j) = i$) and 0 otherwise.

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.
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$$\mathbb{E}[S_i] = \sum_{j=1}^{m} \mathbb{E}[S_{i,j}] = m \cdot \frac{1}{m} = 1$$

$\frac{1}{m}$

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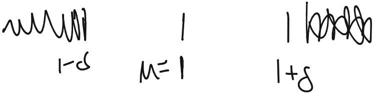

By the Chernoff Bound: for any $\delta \geq 0$,

$$\Pr(\mathbf{S}_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{j=1}^{m} \mathbf{S}_{i,j} - 1\right| \geq \delta \cdot \mu\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

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$\approx 20\log m$

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$\delta = 10 \quad Pr(S_i \geq 11) \leq 2e^{-\frac{10^2}{12}} \approx 2e^{-8}$

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$$\Pr(\mathsf{S}_i \geq 20\log m + 1) \leq 2\exp\left(-\frac{(20\log m)^2}{2+20\log m}\right) \leq 2\exp(-18\log m) \leq \frac{2}{m^{18}}.$$

$(e^{-\log m})^{18} = \frac{1}{m^{18}}$

Apply Union Bound:

$$\Pr(\max_{i\in[m]} \mathsf{S}_i \geq 20\log m + 1) = \Pr\left(\bigcup_{i=1}^{m} (\mathsf{S}_i \geq 20\log m + 1)\right) \leq \sum Pr(S_i \geq 20\log m + 1)$$

$$\leq m \cdot \frac{2}{m^{18}} = \frac{2}{m^{17}}$$

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20 log m + 1

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- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).

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- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a *k-wise independent hash function* for $k = O(\log m)$.

Questions on Exponential Concentration Bounds?

This concludes the probability foundations part of the course – on to algorithms.

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n items

O(n)

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- Allow small probability $\delta > 0$ of false positives. I.e., for any x,

$\delta = .01$

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cuckoo hash

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Solution: Bloom filters (repeated random hashing). Will use much less space than a hash table.

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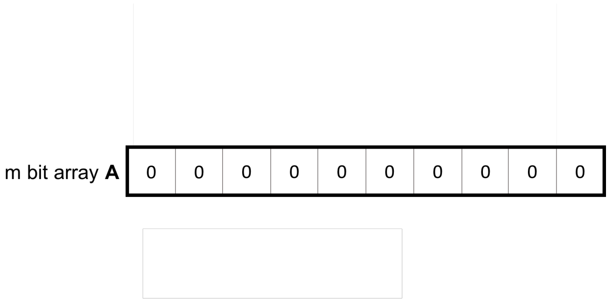

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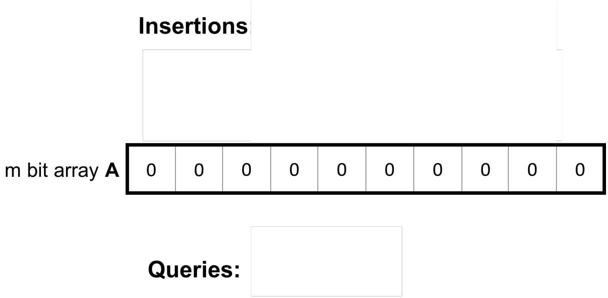

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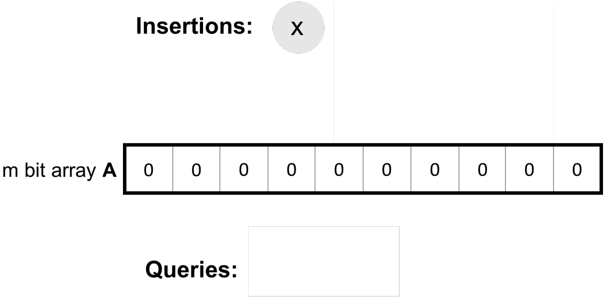

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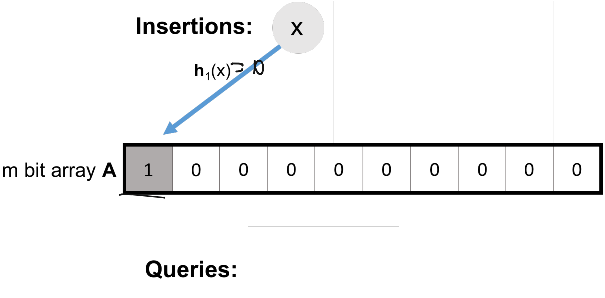

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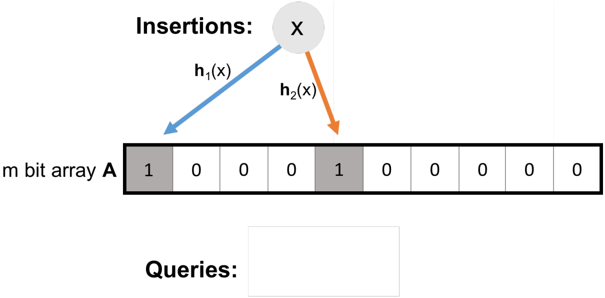

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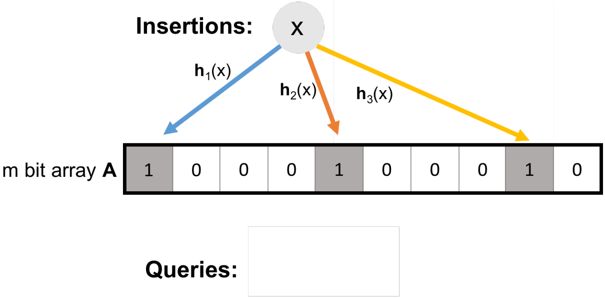

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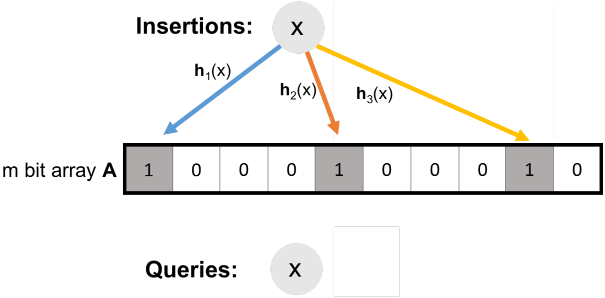

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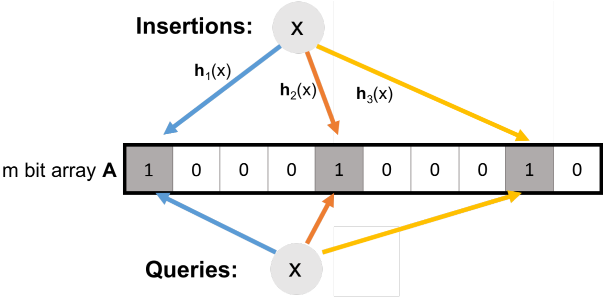

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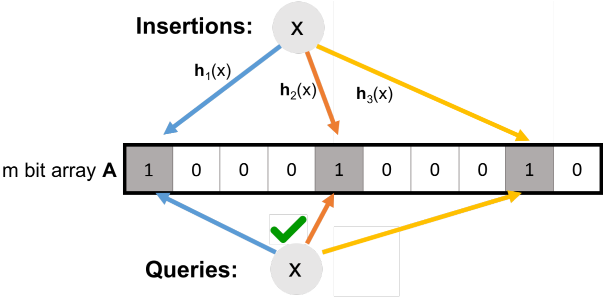

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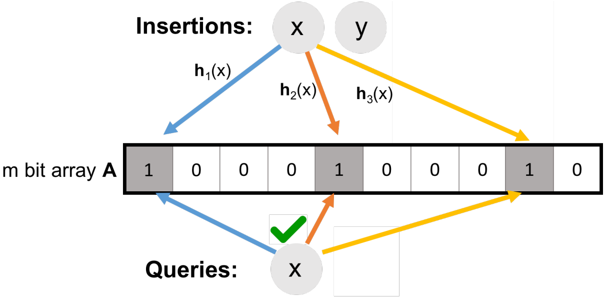

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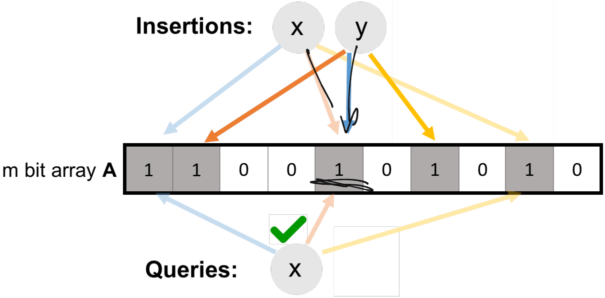

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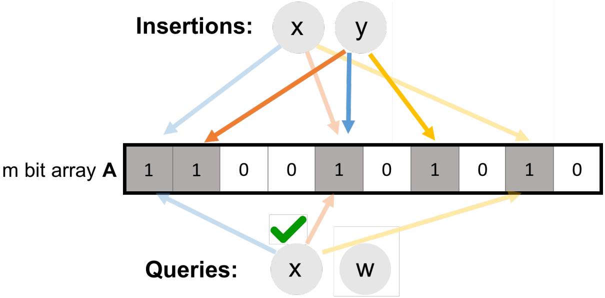

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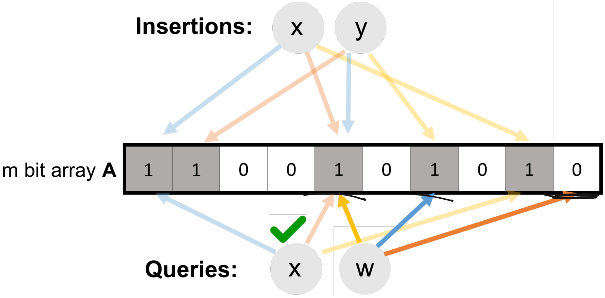

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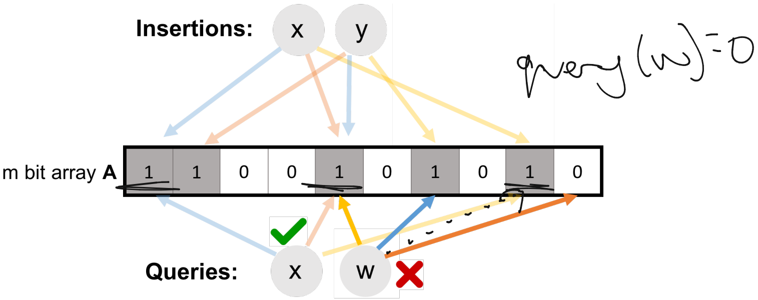

No false negatives. False positives more likely with more insertions.

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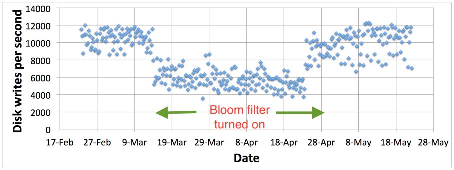

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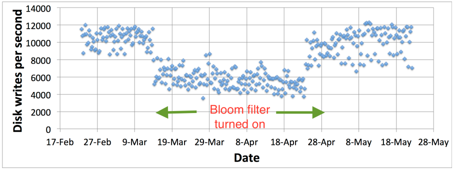

- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.

count-min sketch

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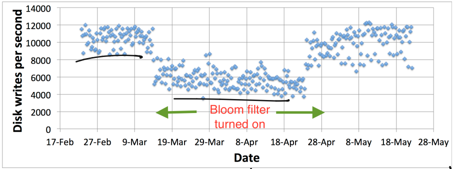

x has never been seen by query(x) = 1 w.p. δ

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- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

APPLICATIONS: DATABASES

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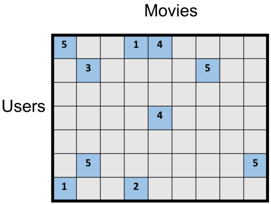

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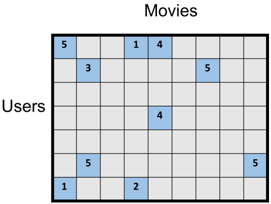

- When a new rating is inserted for $(user_x, movie_y)$, add $(user_x, movie_y)$ to a bloom filter.
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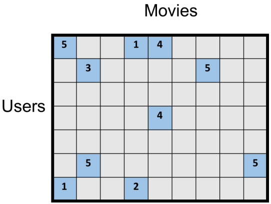

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- **False positive:** A read is made to a possibly empty cell. A $\delta = .05$ false positive rate gives a 95% reduction in these empty reads.

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INNER JOIN

Customers

CustomerId	Name
1	Robert
2	Peter
3	Smith

Orders

OrderId	CustomerId	OrderDate
100	1	2016-10-19 15:21:27
200	4	2016-10-20 15:21:27
300	2	2016-10-21 15:21:27

INNER JOIN on CustomerId Column

RESULT

CustomerId	Name	OrderId	CustomerId	OrderDate
1	Robert	100	1	2016-10-19 15:21:27
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- Matches up a key in column A of one table to a key in column B of another, and merges corresponding information.

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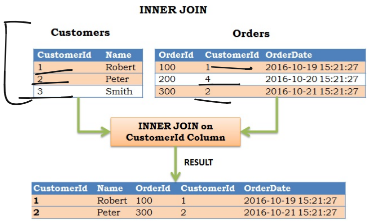

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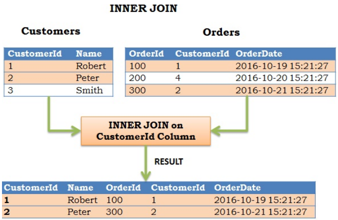

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- A false positive rate of δ means that a $1 - \delta$ fraction of these entries can be eliminated in the initial bloom filter check.

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MORE APPLICATIONS

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- **Digital Currency:** Some Bitcoin clients use bloom filters to quickly pare down the full transaction log to transactions involving bitcoin addresses that are relevant to them (SPV: simplified payment verification).

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$$\Pr(A[i] = 0) = \Pr\left(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_k) \neq i \cap \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\right)$$

n items, k hashes per item

$h_1(x_1) \neq i$ $\quad \left(1 - \frac{1}{m}\right)^{kn}$

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Actually Incorrect! Dependent events.

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- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = \frac{t}{m}$.

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$$\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1)$$
$$= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).$$

I.e., the events $A[\mathbf{h}_1(w)] = 1$,..., $A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\begin{aligned}\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\ = \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).\end{aligned}$$

I.e., the events $A[\mathbf{h}_1(w)] = 1$,..., $A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of: `http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

Questions on Bloom Filters?