COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 3

By Thursday:

- Sign up for Piazza.
- Sign up for Gradescope (code on class website) and fill out the Gradescope consent poll on Piazza. Contact me via email if you don't consent to use Gradescope.

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First Problem Set: released Saturday, due 9/11 at 8pm in Gradescope.

- Remember you can complete in a group of up to 3 students, who all turn in one submission with three names on it.

WEEK 1 QUIZ

91 students completed the quizzes – make sure that if you are enrolled you are doing the quiz each week.

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Question 1: The expected number of inches of rain on Saturday is 2 and the expected number of inches on Sunday is 6. There is a 50% chance of rain on Saturday. If it rains on Saturday, there is a 75% chance of rain on Sunday. If it does not rain on Saturday, there is only a 25% chance of rain on Sunday. What is the expected number of inches of rainfall total over the weekend?

$$\mathbb{E}[R_{sat} + R_{sun}] = \mathbb{E}R_{sat} + \mathbb{E}R_{sun} = 2+6 = 8$$

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Concerns: Probability/linear algebra background, proofs/derivations.

Last Class We Covered:

- Markov's inequality: the most fundamental concentration bound.
- Algorithmic applications of Markov's inequality, linearity of expectation, and indicator random variables:
  - Counting collisions to estimate CAPTCHA database size.
  - Counting collisions to understand the runtime of hash tables with random hash functions.

$$\frac{70^2}{2 \cdot 365} = 7$$

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- Markov's inequality: the most fundamental concentration bound.
- Algorithmic applications of Markov's inequality, linearity of expectation, and indicator random variables:
  - Counting collisions to estimate CAPTCHA database size.
  - Counting collisions to understand the runtime of hash tables with random hash functions.
- Collision counting is closely related to the birthday paradox.

$$\mathbb{E}[C] = \frac{\binom{m}{2}}{n} \approx \frac{m^2}{2n}$$

m = # people in room

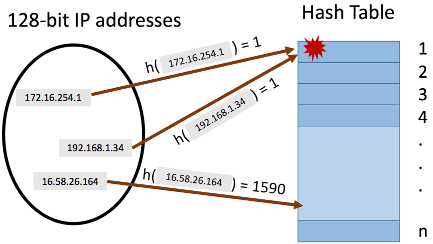
$n = 365$

Today:

- Finish up random hash functions and hash tables.
- See an applications of random hashing to load balancing in distributed systems.
- Through these applications learn about:
  - Chebyshev's inequality, which strengthens Markov's inequality.
  - The union bound, for understanding the probabilities of correlated random events.

We store *m* items from a large universe in a hash table with *n* positions.

insert
deletion
query



- Want to show that when $\mathbf{h} : U \to [n]$ is a random hash function, query time is $O(1)$ with good probability.
- Equivalently: want to show that there are few collisions between hashed items.

When storing m items in a table of size n, the expected number of pairwise collisions (two items stored in the same slots) is:

$$\mathbb{E}[\mathsf{C}] = \frac{m(m-1)}{2n}.$$

m: total number of stored items, n: hash table size, C: total pairwise collisions in table.

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- For $n = 4m^2$ we have: $\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}$.
- By Markov's inequality there no collisions with probability at least $\frac{7}{8}$.

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$O(1)$ query time, but we are using $O(m^2)$ space to store m items...

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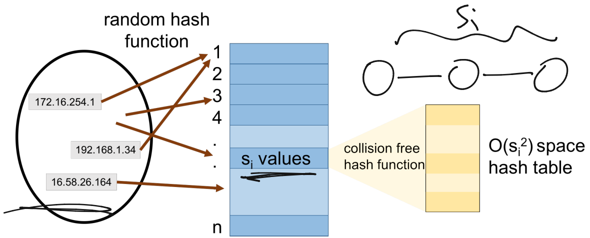

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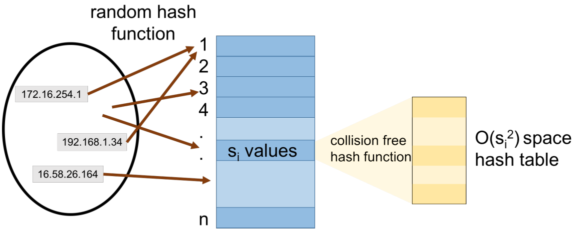

- For each bucket with s_i values, pick a collision free hash function mapping $[s_i] \rightarrow [s_i^2]$.

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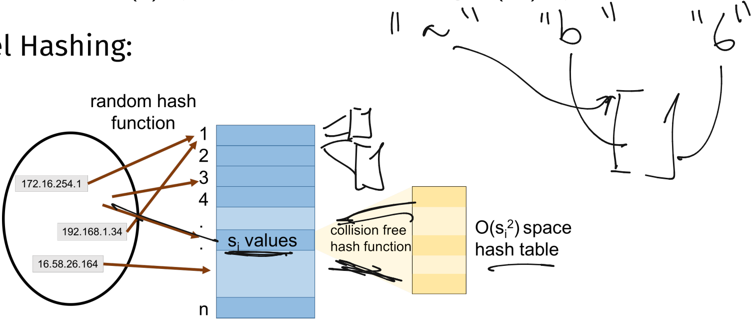

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- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so can just generate a random hash function and check if it is collision free.

SPACE USAGE

Query time for two level hashing is $O(1)$: requires evaluating two hash functions.

x_j, x_k: stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored in hash table at position i.

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"static hashing"

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Collisions again!

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$$(I_1 + I_2 + \ldots I_m)(I_1 + I_2 + \ldots I_m)$$

$$I_1^2 + I_1 I_2 + I_2 \cdot I_1$$

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  $= \mathbb{E}[\mathbb{I}_{h(x_j)=i}]$ //

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X_1 X_2 X_3 X_4

0 1 1 0

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$$= \frac{m}{n} + \frac{m(m-1)}{n^2}$$

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$$= m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2}$$

$$= \frac{m}{n} + \frac{m(m-1)}{n^2} \leq 2 \text{ (If we set } n = m.)$$

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$$1 + \leq 1 \leq 2$$

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x_j, x_k: stored items, m: # stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored at pos i.

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Near optimal space with $O(1)$ query time!

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EFFICIENTLY COMPUTABLE HASH FUNCTION

So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time if we hash m values. Making our whole quest for $O(1)$ query time pointless!

trie

x	**h(x)**
x_1	45
x_2	1004
x_3	10
⋮	⋮
x_m	12

"abc" = 1365

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For fully random hash function: $\frac{1}{n}$

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Efficient Alternative: Let p be a prime with $p \ge |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Let:

$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

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Remember: A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

NEXT STEP

1. We'll consider an application where our toolkit of linearity of expectation + Markov's inequality doesn't give much.

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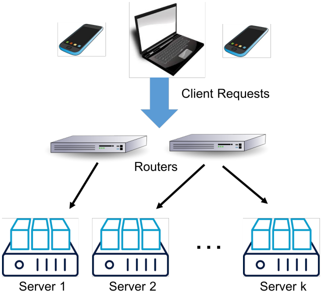
2. Then we'll show how a simple twist on Markov's can give a much stronger result.

Randomized Load Balancing:



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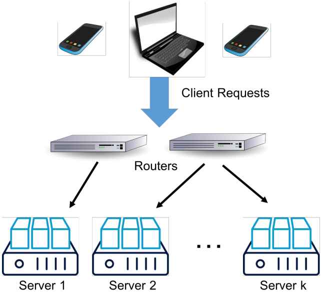

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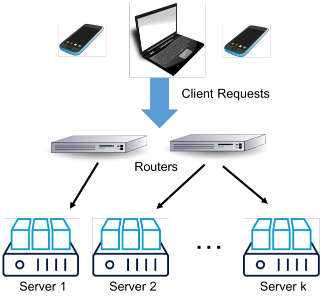

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- Often assignment is done via a random hash function. Why?

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WEAKNESS OF MARKOV'S

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Not great...half the servers may be overloaded.

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