COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 2

By Next Thursday 9/3:

- Sign up for Piazza.
- Sign up for Gradescope (code on course website) and fill out the Gradescope consent poll on Piazza. Contact me via email if you don't consent to use Gradescope.

REMINDER

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By Next Monday 8/31, 8pm:

- Complete Moodle Quiz – posted under Assignments tab on course website.

Last Class We Covered:

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- Basic probability review. See course site for links to resources to refresh your probability background.

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- Linearity of expectation: $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ always.
- Linearity of variance: $\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ if X and Y are independent.

Today:

- An algorithmic application of of linearity of expectation and variance.
- Introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

LINEARITY OF VARIANCE

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$= E[(X - E[X])^2]$

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Together give:

$X^2 + 2XY + Y^2$

$$\text{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$
$$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - (\mathbb{E}[X] + \mathbb{E}[Y])^2$$

(linearity of expectation)

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&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\
&= \text{Var}[X] + \text{Var}[Y].
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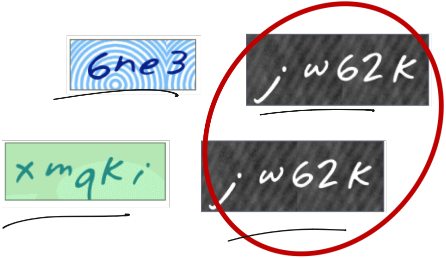
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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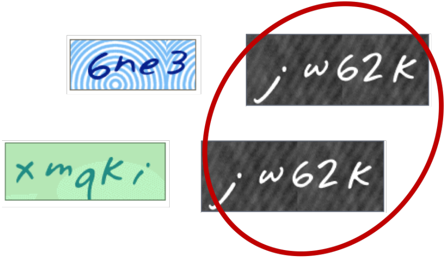

'Mark and recapture' method in ecology.

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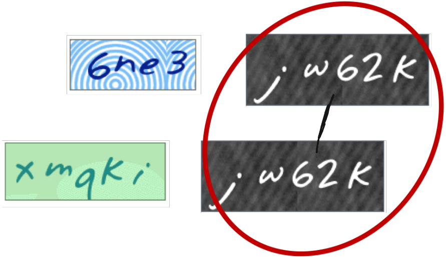

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$n = 1{,}000{,}000$

Breakout: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

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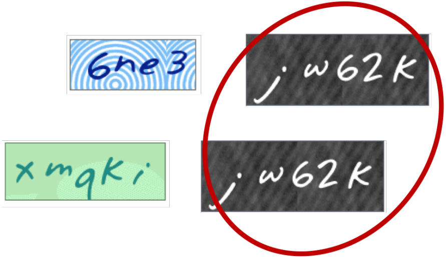

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$$\frac{m(m-1)}{n}$$

Breakout: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i,j), (i,k) and (j,k).

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

n: number of CAPTCHAS in database, m: number of random CAPTCHAS drawn to check database size, $\mathbf{D}$: number of pairwise duplicates in m random CAPTCHAS

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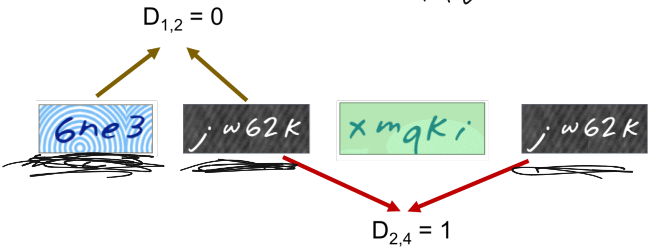
D12 = 1



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$$D = \sum_{i,j \in [m], i \neq j} D_{i,j}.$$

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$$\mathbb{E}[D] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[D_{i,j}].$$

↳ dependent

n different captchas

draw 1 prob. another is the same is $\frac{1}{n}$

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LINEARITY OF EXPECTATION

$m = 3$ (1, 2) (1, 3) (2, 3)

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$$\mathbb{E}[\mathbf{D}] = \sum_{i,j \in [m], i \neq j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

$\left(\frac{1}{n} \cdot (\#\text{ pairs } i, j \in [m], i \neq j)\right)$

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Note that the $\mathrm{D}_{i,j}$ random variables are not independent!

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You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

1000 · 999

$$\mathbb{E}[\mathbf{D}] = \frac{m(m-1)}{2n} = .4995$$

2,000,000

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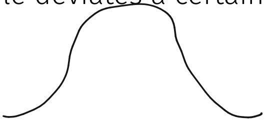
Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.



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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The most fundamental concentration bound: **Markov's inequality.**

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non-negative because X is non-negative

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For any non-negative random variable X and any $t > 0$:

$$\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}.$$

$t \to t \cdot \mathbb{E}[X]$

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The larger the deviation t, the smaller the probability.

Expected number of duplicate CAPTCHAS:

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You see $D = 10$ duplicates.

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BACK TO OUR APPLICATION

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.0005

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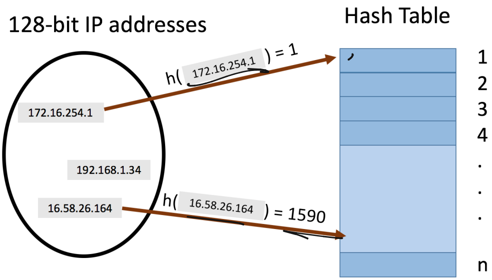
Classic Solution: Hash tables

- *Static hashing* since we won't worry about insertion and deletion today.



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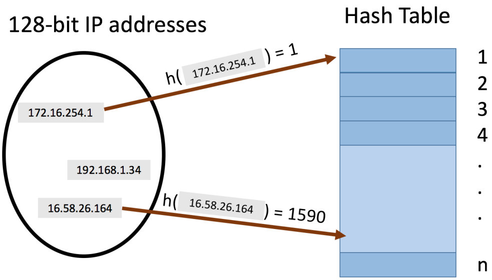

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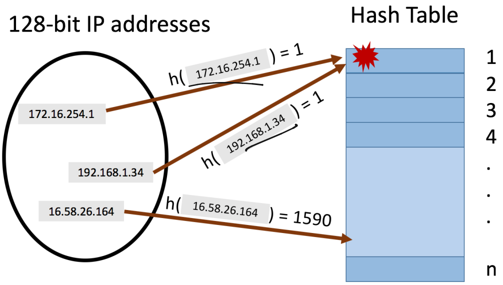

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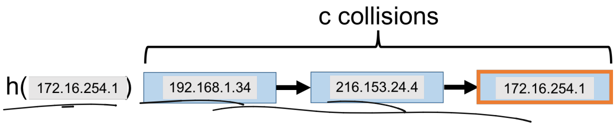
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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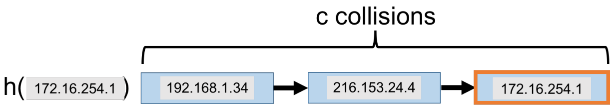

How Can We Bound c?

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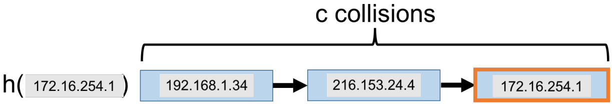

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- In the worst case could have $c = m$ (all items hash to the same location).

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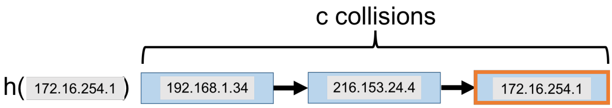

How Can We Bound c?

- In the worst case could have $c = m$ (all items hash to the same location).
- Two approaches: 1) we assume the items inserted are chosen randomly from the universe U or 2) we assume the hash function is random.

Let $\mathrm{h} : U \to [n]$ be a random hash function.

- I.e., for $x \in U$, $\Pr(\mathrm{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathrm{h}(x), \mathrm{h}(y)$ are independent for any two items $x \neq y$.

$h(127.0.01) = 1, 2, 3, \ldots, n$

$h(x), \quad h(x)$

RANDOM HASH FUNCTION

$h^{-1}(x)$ $h(x) = 15$
wait 10 minute
$h(x) = 15$

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

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Short Breakout: Assuming we insert m elements into a hash table of size n, what is the expected total number of pairwise collisions?

Let $\mathsf{C}_{i,j} = 1$ if items i and j collide $(\mathsf{h}(x_i) = \mathsf{h}(x_j))$, and 0 otherwise. The number of pairwise duplicates is:

$$\mathsf{C} = \sum_{i,j \in [m], i \neq j} \mathsf{C}_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, h: random hash function.

Let $C_{i,j} = 1$ if items i and j collide ($h(x_i) = h(x_j)$), and 0 otherwise. The number of pairwise duplicates is:

$$\mathbb{E}[C] = \sum_{i,j \in [m], i \neq j} \mathbb{E}[C_{i,j}]. \qquad \text{(linearity of expectation)}$$

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$$\mathbb{E}[\mathsf{C}_{i,j}] = \Pr[\mathsf{C}_{i,j} = 1] = \Pr[\mathbf{h}(x_i) = \mathbf{h}(x_j)]$$

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Identical to the CAPTCHA analysis!

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, $\mathbf{h}$: random hash function.

items

$$\mathbb{E}[C] = \frac{m(m-1)}{2n}.$$

tables

m: total number of stored items, *n*: hash table size, C: total pairwise collisions in table.

COLLISION FREE HASHING

$$\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{2n}.$$

- For $n = 4m^2$ we have: $\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}$.

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- ~~**Breakout:**~~ Give a lower bound on the probability that we have no collisions, i.e., $\Pr[\mathbf{C} = 0]$?

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Apply Markov's Inequality:

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Apply Markov's Inequality: $\Pr[C \geq 1] \leq \frac{\mathbb{E}[C]}{1}$

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m: total number of stored items, n: hash table size, C: total pairwise collisions in table.

$n = 10m^2$

$4m^2$

$$\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{2n}.$$

$O(1)$

$O(m)$

- For $n = 4m^2$ we have: $\mathbb{E}[\mathbf{C}] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}$. $\frac{1}{20}$
- **Breakout:** Give a lower bound on the probability that we have no collisions, i.e., $\Pr[\mathbf{C} = 0]$?

Apply Markov's Inequality: $\Pr[\mathbf{C} \geq 1] \leq \frac{\mathbb{E}[\mathbf{C}]}{1} = \frac{1}{8}$.

$$\Pr[\mathbf{C} = 0] = 1 - \Pr[\mathbf{C} \geq 1] \geq 1 - \frac{1}{8} = \frac{7}{8}.$$

$1 - \frac{1}{20} = 95\%$

Pretty good...but we are using $O(m^2)$ space to store m items...

m: total number of stored items, n: hash table size, $\mathbf{C}$: total pairwise collisions in table.