COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco

University of Massachusetts Amherst. Fall 2020.

Lecture 17

LOGISTICS

- Problem Set 3 deadline extended until Monday 10/26, 8pm.
- Week 9 Quiz will now be due Tuesday 10/27, 8pm.

Last Few Classes: Low-Rank Approximation and PCA

- Compress data that lies close to a k-dimensional subspace.
- Equivalent to finding a low-rank approximation of the data matrix $\mathbf{X}$: $\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T$ for orthonormal $\mathbf{V} \in \mathbb{R}^{d \times k}$.
- Optimal solution via PCA (eigendecomposition of $\mathbf{X}^T\mathbf{X}$ or equivalently, SVD of $\mathbf{X}$).
- Singular vectors of $\mathbf{X}$ are the eigenvectors of $\mathbf{X}\mathbf{X}^T$ and $\mathbf{X}^T\mathbf{X}$. Singular values squared are the eigenvalues.

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This Class: Applications of low-rank approx. beyond compression.

- Matrix completion and collaborative filtering
- Entity embeddings (word embeddings, node embeddings, etc.)
- Low-rank approximation fornon-linear dimensionality reduction.
- Spectral graph theory, spectral clustering.

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X

Movies

Users

5	3	3	1	4	4	4	3	5
4	3	3	1	4	4	5	3	5
3	3	3	2	3	3	3	3	3
4	3	3	4	4	4	4	3	3
3	3	3	2	3	3	3	3	3
2	5	3	4	4	4	4	4	5
1	3	3	2	3	3	3	1	2

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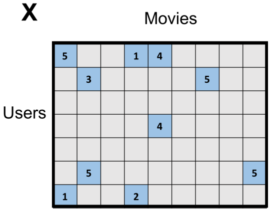

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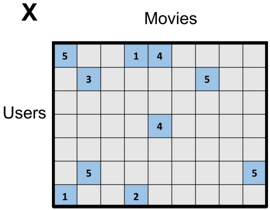

Solve: $Y = \underset{\text{rank }-k\ \mathbf{B}}{\arg\min} \sum_{\text{observed }(j,k)} \left[\mathbf{X}_{j,k} - \mathbf{B}_{j,k}\right]^2$

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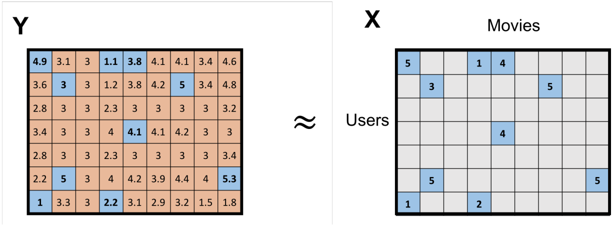

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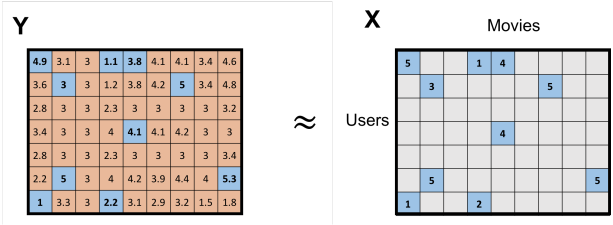

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Y

4.9	3.1	3	**1.1**	**3.8**	4.1	4.1	3.4	4.6
3.6	**3**	3	1.2	3.8	4.2	**5**	3.4	4.8
2.8	3	3	2.3	3	3	3	3	3.2
3.4	3	3	4	**4.1**	4.1	4.2	3	3
2.8	3	3	2.3	3	3	3	3	3.4
2.2	**5**	3	4	4.2	3.9	4.4	4	**5.3**
1	3.3	3	**2.2**	3.1	2.9	3.2	1.5	1.8

$\approx$

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- Words (to identify synonyms, translations, etc.)
- Nodes in a social network

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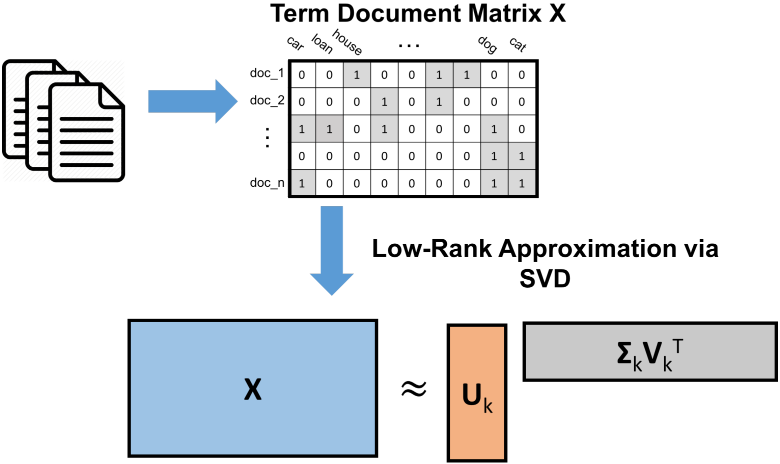
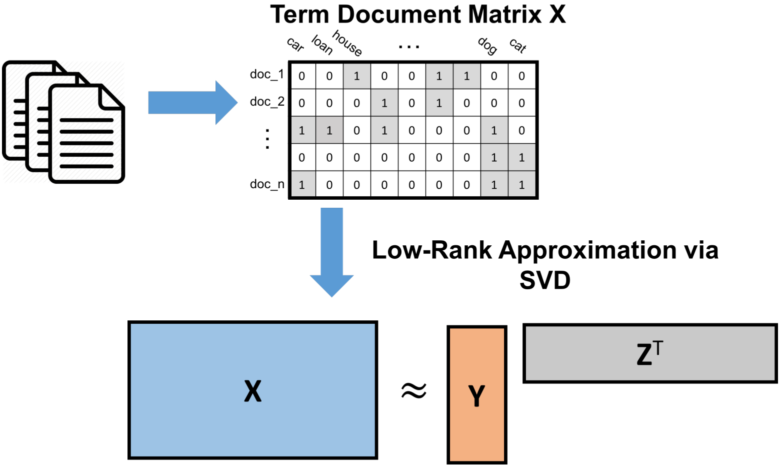
Usual Approach: Convert each item into a high-dimensional feature vector and then apply low-rank approximation.

EXAMPLE: LATENT SEMANTIC ANALYSIS

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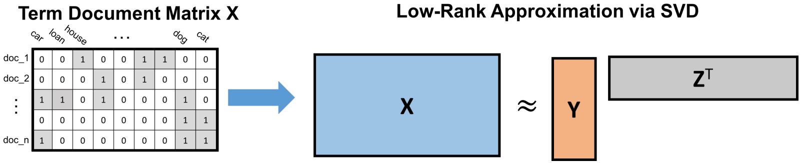
car loan house . . . dog cat
doc_1 0 0 1 0 0 1 1 0 0
doc_2 0 0 0 1 0 1 0 0 0
⋮
1 1 0 1 0 0 0 1 0
0 0 0 0 0 0 0 1 1
doc_n 1 0 0 0 0 0 0 1 1
Low-Rank Approximation via SVD
X
≈
Y
Z^T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Low-Rank Approximation via SVD
X ≈ Y Z^T

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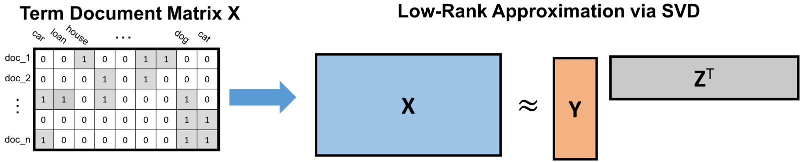

- If the error $\|\mathbf{X} - \mathbf{Y}\mathbf{Z}^T\|_F$ is small, then on average,

$$\mathbf{X}_{i,a} \approx (\mathbf{Y}\mathbf{Z}^T)_{i,a} = \langle \vec{y}_i, \vec{z}_a \rangle.$$

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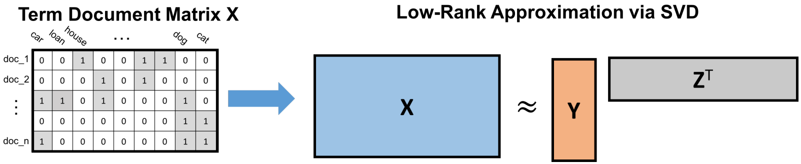

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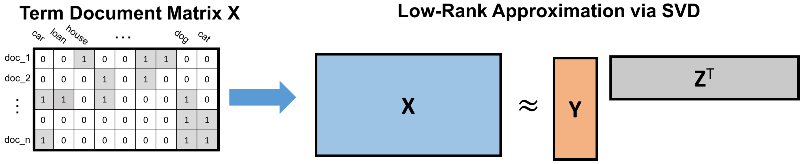

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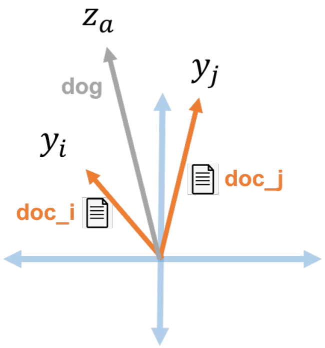

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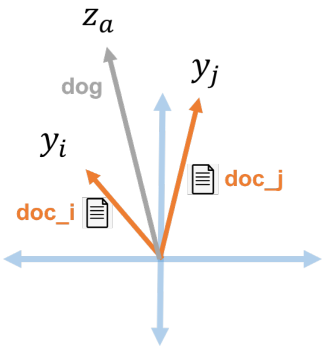

Another View: Each column of **Y** represents a 'topic'. $\vec{y}_i(j)$ indicates how much doc_i belongs to topic j. $\vec{z}_a(j)$ indicates how much $word_a$ associates with that topic.

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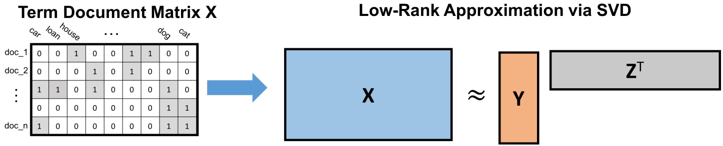

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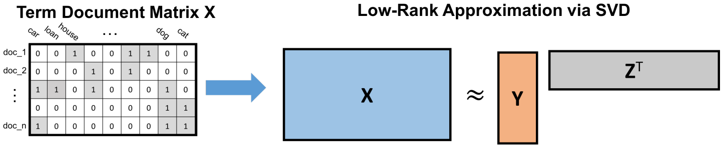

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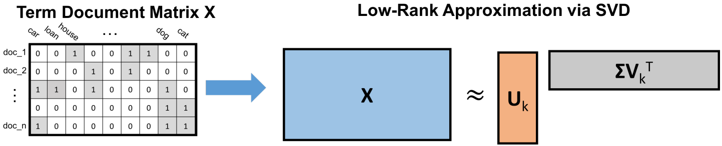

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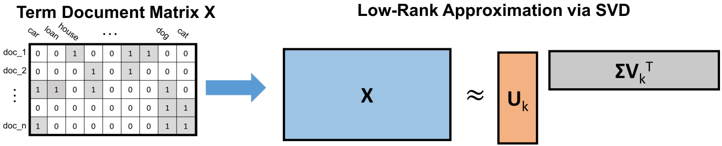

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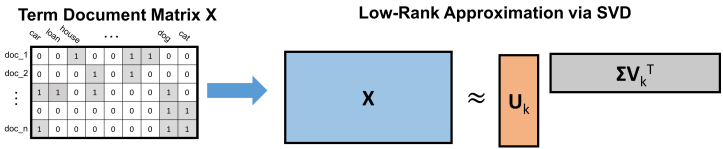

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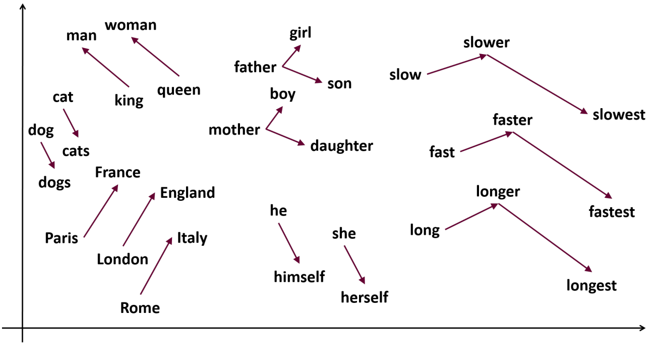
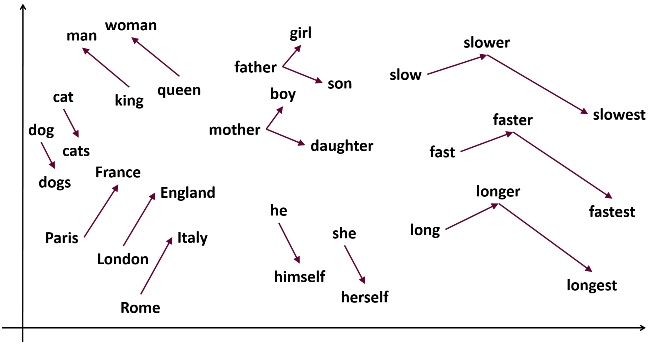
- Many ways to measure similarity: number of sentences both occur in, number of times both appear in the same window of w words, in similar positions of documents in different languages, etc.
- Replacing $\mathbf{X}^T\mathbf{X}$ with these different metrics (sometimes appropriately transformed) leads to popular word embedding algorithms: word2vec, GloVe, fastText, etc.

EXAMPLE: WORD EMBEDDING





Note: word2vec is typically described as a neural-network method, but it is really just low-rank approximation of a specific similarity matrix. *Neural word embedding as implicit matrix factorization*, Levy and Goldberg.