# COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

Cameron Musco
University of Massachusetts Amherst. Spring 2020.
Lecture 10

# LOGISTICS

- Problem Set 2 is due Sunday 3/8.
- Midterm on Thursday, 3/12. Will cover material through today.
- I ~~have~~ will posted a study guide and practice questions on the course schedule.
- Next Tuesday I can't do office hours after class. I will hold them before class on Tuesday (10:00am - 11:15am) and after class on Thursday (12:45pm-2:00pm).

Last Class: Dimensionality Reduction

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- Finished up Count-Min Sketch and Frequent Items.
- Applications and examples of dimensionality reduction in data science (PCA, LSA, autoencoders, etc.)
- Low-distortion embeddings and some simple cases of when no-distortion embeddings are possible.

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- Low-distortion embeddings and some simple cases of when no-distortion embeddings are possible.

## The Johnson-Lindenstrauss Lemma.

- Any data set can be embedded with low distortion into low-dimensional space.
- Prove the JL Lemma.
- Discuss algorithmic considerations, connections to other methods (SimHash), etc.

**Low Distortion Embedding:** Given $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$, distance function $D$, and error parameter $\epsilon \geq 0$, find $\tilde{x}_1, \ldots, \tilde{x}_n \in \mathbb{R}^m$ (where $m \ll d$) and distance function $\tilde{D}$ such that for all $i, j \in [n]$:

$$(1-\epsilon)D(\vec{x}_i, \vec{x}_j) \leq \tilde{D}(\tilde{x}_i, \tilde{x}_j) \leq (1+\epsilon)D(\vec{x}_i, \vec{x}_j).$$

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We will primarily focus on this restricted notion in this class.

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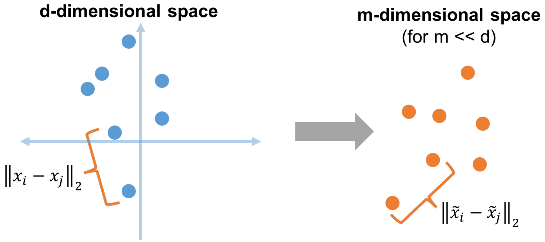


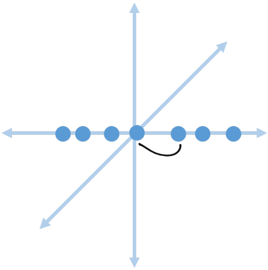
# EMBEDDING WITH ASSUMPTIONS

Assume that $\vec{x}_1, \ldots, \vec{x}_n$ all lie on the $1^{st}$ axis in $\mathbb{R}^d$.

$[x_i(1), 0] \; ; x_i$

$\downarrow$

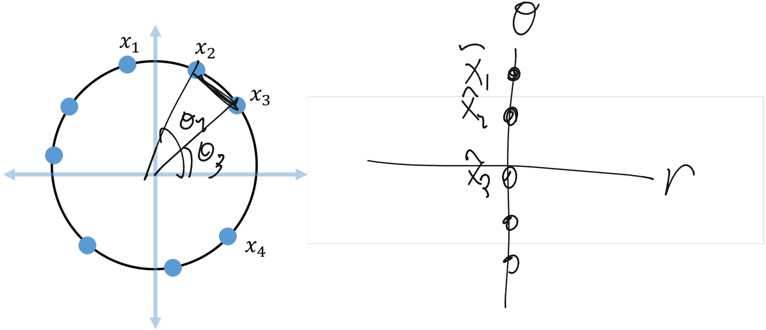
$[x_i(1)] \; ; \hat{x}_i$



Set $m = 1$ and $\tilde{x}_i = \vec{x}_i(1)$ (i.e., $\tilde{x}_i$ is just a single number).

- $\|\tilde{x}_i - \tilde{x}_j\|_2 = \sqrt{[\vec{x}_i(1) - \vec{x}_j(1)]^2} = |\vec{x}_i(1) - \vec{x}_j(1)| = \|\vec{x}_i - \vec{x}_j\|_2$.
- An embedding with no distortion from any $d$ into $m = 1$.

Assume that $\vec{x}_1, \ldots, \vec{x}_n$ all lie on the unit circle in $\mathbb{R}^2$.

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- Does it admit a low-distortion Euclidean embedding?

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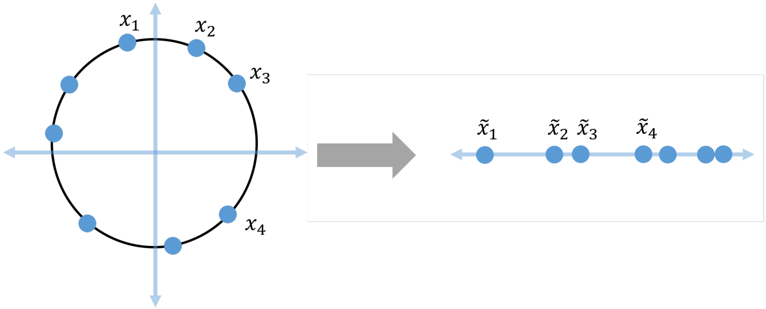


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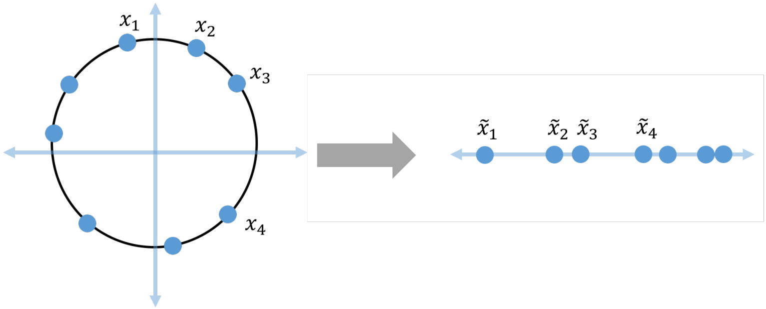
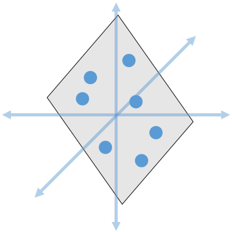



- Admits a low-distortion embedding to 1 dimension by letting $\tilde{x}_i = \theta(\vec{x}_i)$.
- Does it admit a low-distortion Euclidean embedding? No! Send me a proof on Piazza for 3 bonus points on Problem Set 2.

**Another easy case:** Assume that $\vec{x}_1, \ldots, \vec{x}_n$ lie in any $k$-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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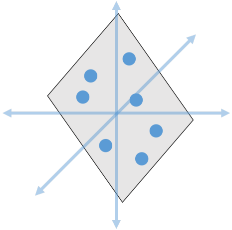

- Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and let $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns.

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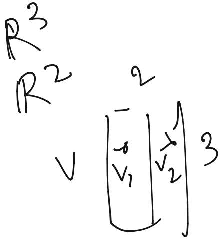

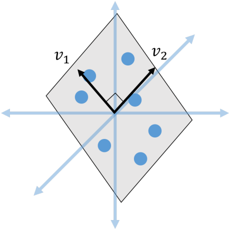


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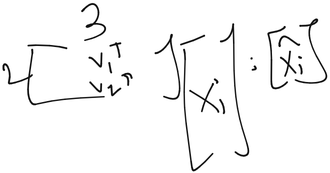

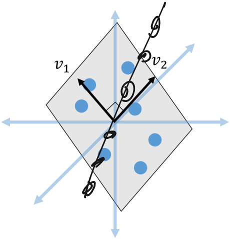


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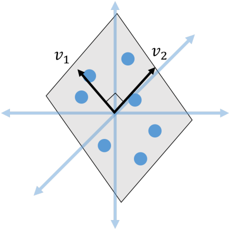


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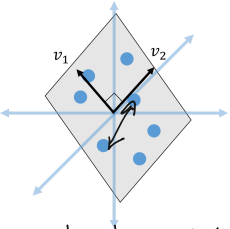


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- An embedding with no distortion from any $d$ into $m = k$.
- $\mathbf{V}^T : \mathbb{R}^d \to \mathbb{R}^k$ is a linear map giving our embedding.

What about when we don't make any assumptions on $\vec{x}_1, \ldots, \vec{x}_n$. I.e., they can be scattered arbitrarily around $d$-dimensional space?

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- Can we find an $\epsilon$-distortion embedding into $m \ll d$ dimensions for $\epsilon > 0$?

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- Can we find a no-distortion embedding into $m \ll d$ dimensions? No. Require $m = d$.
- Can we find an $\epsilon$-distortion embedding into $m \ll d$ dimensions for $\epsilon > 0$? Yes! Always, with $m$ depending on $\epsilon$.

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## THE JOHNSON-LINDENSTRAUSS LEMMA

$V^T$

**Johnson-Lindenstrauss Lemma:** For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\mathbf{\Pi} : \mathbb{R}^d \to R^m$ such that $m = O\left(\frac{\log n}{\epsilon^2}\right)$ and letting $\tilde{x}_i = \mathbf{\Pi}\vec{x}_i$:

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Further, if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ has each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, it satisfies the guarantee with high probability.

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For $d = 1$ trillion, $\epsilon = .05$, and $n = 100,000$, $m \approx 6600$.

[.5 .6 1 0.7]    $n = 2^d$

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Very surprising! Powerful result with a simple construction: applying a random linear transformation to a set of points preserves distances between all those points with high probability.

For any $\vec{x}_1, \ldots, \vec{x}_n$ and $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ with each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, with high probability, letting $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$:

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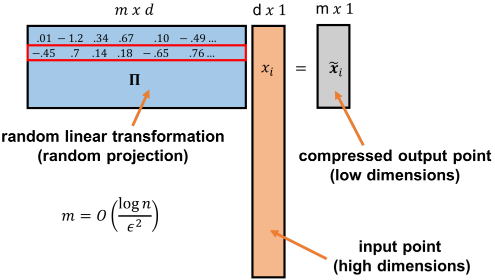

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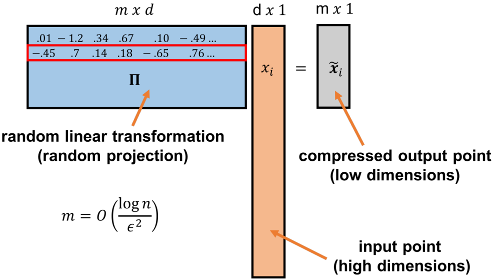


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# RANDOM PROJECTION

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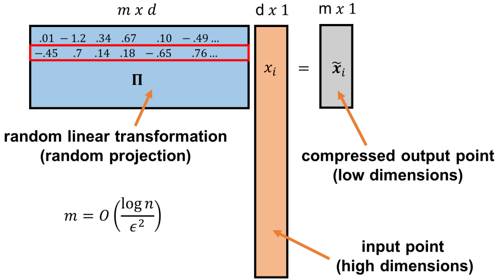


- $\mathbf{\Pi}$ is known as a random projection. It is a random linear function, mapping length $d$ vectors to length $m$ vectors.
- $\mathbf{\Pi}$ is data oblivious. Stark contrast to methods like PCA.

- Many alternative constructions: $\pm 1$ entries, sparse (most entries 0), Fourier structured (Problem Set 2), etc. $\implies$ more efficient computation of $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$.

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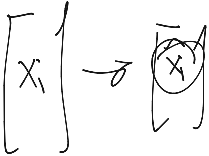
- Memory needed is just $O(d + nm)$ vs. $O(nd)$ to store the full data set.
- Compression can also be easily performed in parallel on different servers.
- When new data points are added, can be easily compressed, without updating existing points.

Compression operation is $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$, so for any $j$,



$$\tilde{\mathbf{x}}_i(j) = \langle \mathbf{\Pi}(j), \vec{x}_i \rangle = \sum_{k=1}^{d} \mathbf{\Pi}(j,k) \cdot \vec{x}_i(k).$$

$\vec{x}_1, \ldots, \vec{x}_n$: original points ($d$ dims.), $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$: compressed points ($m < d$ dims.), $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection (embedding function)

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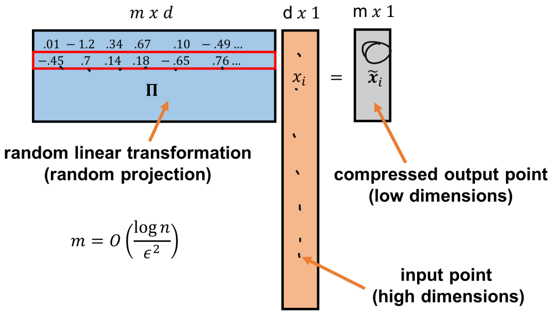


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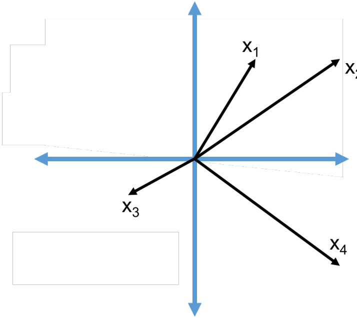


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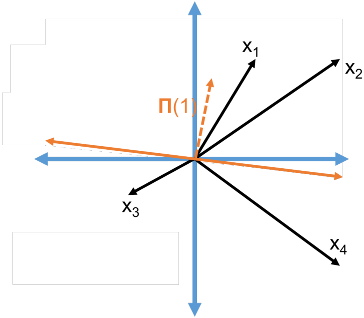


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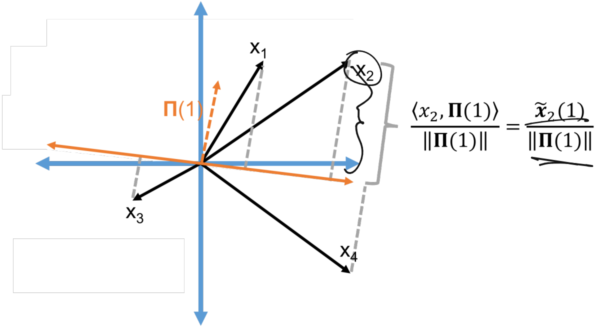


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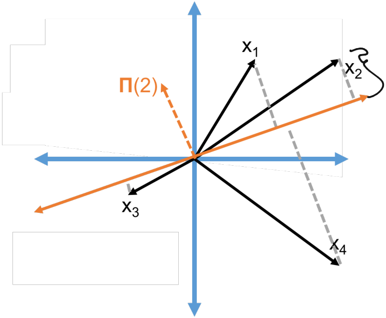
$$\tilde{\mathbf{x}}_i(j) = \langle \mathbf{\Pi}(j), \vec{x}_i \rangle = \sum_{k=1}^{d} \mathbf{\Pi}(j,k) \cdot \vec{x}_i(k).$$

$\mathbf{\Pi}(j)$ is a vector with independent random Gaussian entries.

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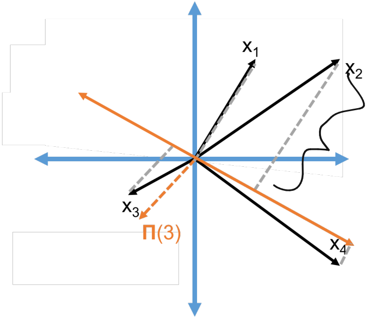


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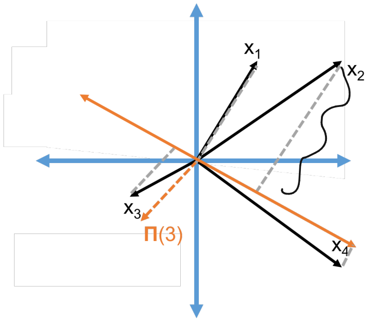


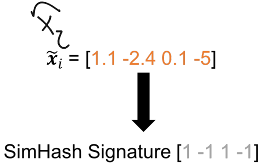


Points with high cosine similarity have similar random projections.

Computing a length $m$ SimHash signature $SH_1(\vec{x}_i), \ldots, SH_m(\vec{x}_i)$ is identical to computing $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$ and then taking $sign(\tilde{\mathbf{x}}_i)$.

The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

**Distributional JL Lemma:** Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

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- Can be proven from first principles. Will see next.

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# DISTRIBUTIONAL JL $\implies$ JL

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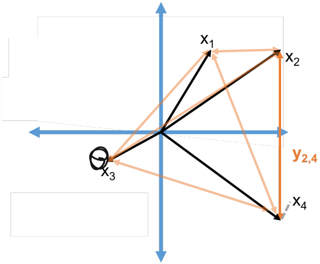


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$$\|\Pi x_i - \Pi x_j\|$$

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$O\left(\frac{\log n}{\epsilon^2}\right)$

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$\varepsilon$ = error
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$$\Pr\left(\|\hat{x}_i - \hat{x}_j\| \not\approx \|x_i - x_j\|\right) \leq \delta'$$

$$\Pr\left(\text{at least one } \|\hat{x}_i - \hat{x}_j\| \not\approx \|x_i - x_j\|\right) \leq \binom{n}{2}\delta'$$

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$= 1 - \delta$

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Yields the JL lemma.

**Distributional JL Lemma:** Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

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# DISTRIBUTIONAL JL PROOF

**Distributional JL Lemma:** Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

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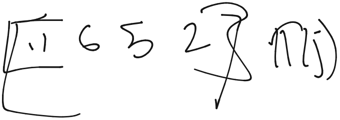

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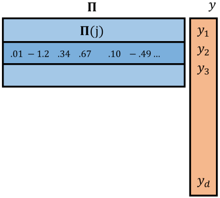


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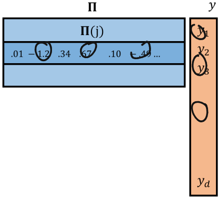


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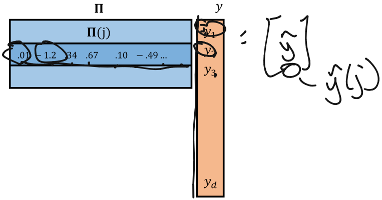


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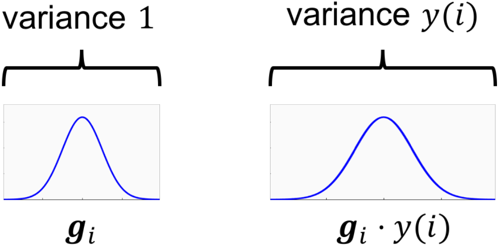
$\sum_{i=1}^{d} N(0, \tilde{y}(i)^2)$

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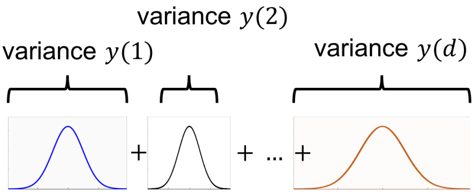


$$\tilde{\mathbf{y}}(j) = \frac{1}{\sqrt{m}}[\boldsymbol{g}_1 \cdot y(1) + \boldsymbol{g}_2 \cdot y(2) + \ldots + \boldsymbol{g}_d \cdot y(d)]$$

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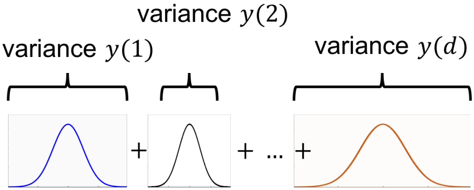


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What is the distribution of $\tilde{\mathbf{y}}(j)$?

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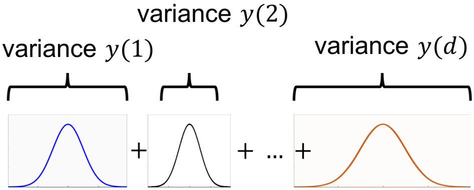


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What is the distribution of $\tilde{\mathbf{y}}(j)$? Also Gaussian!

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Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

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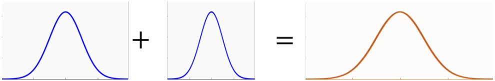

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Stability is another explanation for the central limit theorem.

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**So far:** Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

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What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

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How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

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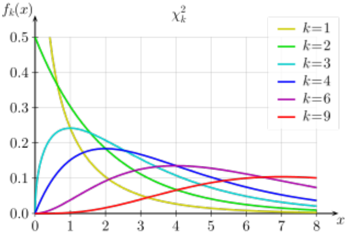


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**Lemma:** (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with $m$ degrees of freedom,

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Gives the distributional JL Lemma and thus the classic JL Lemma!

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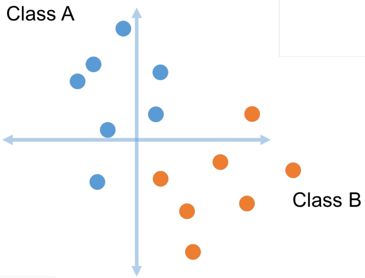

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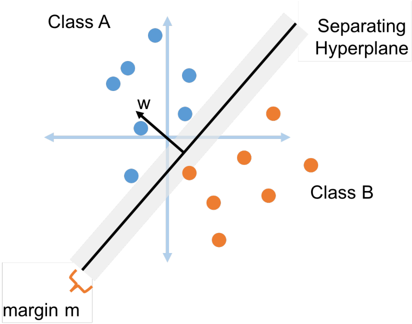

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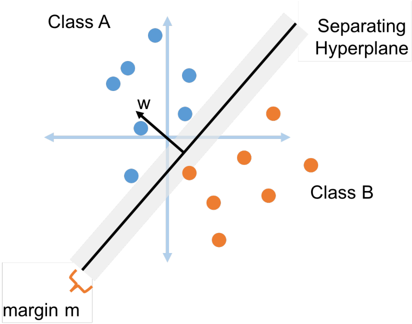


- For any point $a$ in A, $\langle a, w \rangle \geq c + m$
- For any point $b$ in B $\langle b, w \rangle \leq c - m$.
- Assume all vectors have unit norm.

# EXAMPLE APPLICATION: SVM

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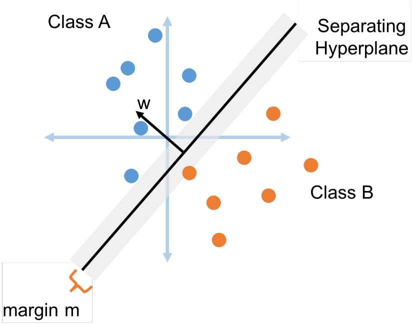


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JL Lemma implies that after projection into $O\left(\frac{\log n}{m^2}\right)$ dimensions, still have $\langle \tilde{\mathbf{a}}, \tilde{\mathbf{w}} \rangle \geq c + m/4$ and $\langle \tilde{\mathbf{b}}, \tilde{\mathbf{w}} \rangle \leq c - m/4$.

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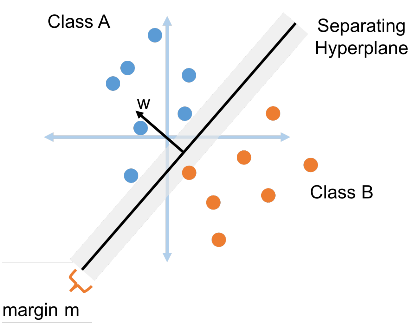


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**Upshot:** Can random project and run SVM (much more efficiently) in the lower dimensional space to find separator $\tilde{\mathbf{w}}$.

Questions?