# The Four Color Theorem

Yuriy Brun

**Abstract.** In this paper, we introduce graph theory, and discuss the Four Color Theorem. Then we prove several theorems, including Euler's formula and the Five Color Theorem.

**1. Introduction.** According to [**1**, p. 19], the Four Color Theorem has fascinated people for almost a century and a half. It dates back to 1852 when Francis Guthrie, while trying to color the map of counties of England, noticed that four colors suffice. He asked his brother Frederick if any map can be colored using four colors so that different colors appear on adjacent regions — that is, regions sharing a common boundary segment, not just a point. Frederick Guthrie then explained the problem to August DeMorgan, who in turn showed it to Arthur Cayley. The problem was first published as a puzzle for the public by Cayley in 1878.

A year later, in 1879, Alfred Kempe published the first proof, which was not disputed until 1890. In 1880, Peter Tait published a different proof, which was proved incomplete in 1891 by Julius Petersen. Both false proofs did have some value, though. Kempe discovered what became known as "Kempe chains," and Tait found an equivalent formulation of the Four Color Theorem in terms of three-edge coloring.

The next major contribution came from George Birkhoff. His work allowed Philip Franklin in 1922 to prove that the four-color conjecture is true for maps with at most twenty-five faces. Other mathematicians made progress using Birkhoff's work. Heinrich Heesch developed the two main ideas needed for the ultimate proof: "reducibility" and "discharging." Although other researchers studied reducibility as well, Heesch was the first to take a close look at the idea of discharging, crucial for the "unavoidability part" of the proof (see below). Heesch also conjectured that these methods can be used to prove the Four Color Theorem, but never actually attempted it himself.

In 1976, Kenneth Appel and Wolfgang Haken [**2**] published their proof of the Four Color Theorem. Their proof has not been challenged to this day. However, it relies heavily on the use of a computer: some parts of it are too lengthy to be verified by hand, and others are so complex that no one has yet tried.

Recently, four mathematicians at Ohio State University and Georgia Institute of Technology — Neil Robertson, Daniel P. Sanders, Paul D. Seymour and Robin Thomas — in [**4**, p. 432], gave a four-coloring algorithm for planar graphs.

The basic idea of their proof is the same as Appel and Haken's. They found a set of 633 different types of graphs they called "configurations," and proved that each of them is irreducible that is, none can appear in a smaller counterexample. Thus eventually they proved that no counterexample exists.

It has been known since 1913 that every minimal counterexample to the Four Color Theorem is an internally six-connected triangulation. In the second part of the proof, published in [**4**, p. 432], Robertson et al. proved that at least one of the 633 configurations appears in every internally six-connected planar triangulation. This condition is called "unavoidability," and uses the discharging method, first suggested by Heesch. Here, the proof differs from that of Appel and Haken in that it relies far less on computer calculation. Nevertheless, parts of the proof still cannot be verified by a human. The search continues for a computer-free proof of the Four Color Theorem.

This paper introduces the basic graph theory required to understand the Four Color Theorem. Section 2 explains the terminology and definitions. Section 3 proves Euler's formula. Section 4 proves several theorems, including the Five Color Theorem, which provide a solid basis for the spirit of the proof of the Four Color Theorem.

**2. Background.** To understand the principles of the Four Color Theorem, we must know some basic graph theory.

A *graph* is a pair of sets, whose elements called *vertices* and *edges* respectively. Associated to each edge are two distinguished vertices called *ends*. The two ends are allowed to coincide; if they do, the edge is called a *loop*. Each vertex is represented by a point in the plane. Each edge is represented by a continuous curve between its two ends. We say an edge *connects* its two ends. Figure 2-1 shows a graph with vertices

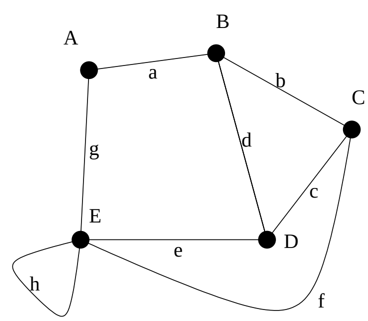


**Figure 2-1.** A graph containing five vertices and eight edges.

$A$, $B$, $C$, $D$, $E$ and edges $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$. The edge $h$ connects the vertex $E$ to itself; so $h$ is a loop.

Two graphs are considered to be the *same* if there is a one-to-one correspondence between their vertices and edges, preserving the ends. Figure 2-2 shows two graphs that

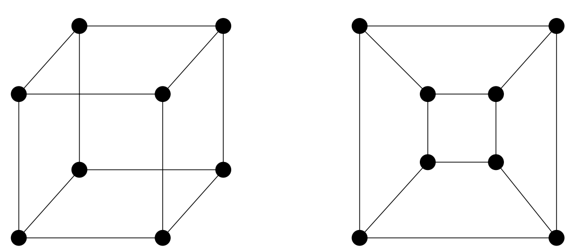

**Figure 2-2.** Two identical cubic graphs. The two representations depict the same graph.

do not look the same, but are in fact the same. These graphs also happen to be *cubic* graphs; that is, they represent the vertices and edges of a $n$-dimensional cube for some integer $n$.

A *path* from a vertex $V$ to a vertex $W$ is a sequence of edges $e_1, e_2, \ldots, e_n$, such that if $V_i$ and $W_i$ denote the ends of $e_i$, then $V_1 = V$ and $W_n = W$ and $W_i = V_{i+1}$ for $1 \leq i < n$. A *cycle* is a path that involves no edge more than once and $V = W$. Any of the vertices along the path can serve as the initial vertex. For example, a loop is a cycle. A triangle, a square, and all other polygons are also cycles. Figure 2-1 shows several cycles, including the triangle $C \to D \to E \to C$.

There are many types of graphs. In Figure 2-2, we encountered a cubic graph. Another type is a tree: a *tree* is a graph with no cycles. Some graphs are connected, and some are disconnected. By definition, a graph is *disconnected* if there exist at least two vertices between which there is no path. A graph that is not disconnected is called *connected*.

A *directed* graph is a graph in which one of each edge's ends is marked as "start" and the other as "finish". An example is shown in Figure 2-3.

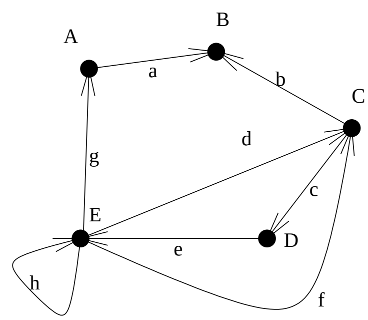


**Figure 2-3.** A sample directed graph.

The *degree* of a vertex is defined as the number of edges with that vertex as an end. A loop contributes twice to the degree of its end. In Figure 2-2, all the vertices are of degree 3.

The *complete* graph on $n$ vertices, denoted $K_n$, has one and only one edge between any pair of distinct vertices. For example, $K_5$ is illustrated in Figure 2-4A.

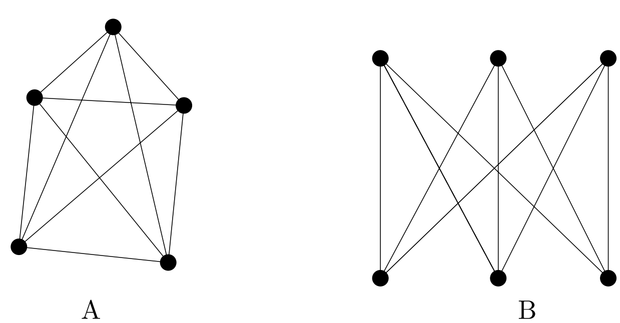


**Figure 2-4.** The complete graph $K_5$ on five vertices A and the complete bipartite graph $K_{3,3}$ on six vertices B.

A *complete bipartite* graph contains two disjoint sets of vertices; an edge connects each of the vertices in one set to each of the vertices in the other, and there are no other edges. A complete bipartite graph on six vertices is shown in Figure 2-4B.

**3. Euler's Formula.** From now on, we will ignore all graphs that have loops or more than one edge connecting any two vertices.

*Planar* graphs are graphs that can be represented in the plane without any edges intersecting away from the vertices. Since a graph may have many different representations, some of them may have edge intersections, but others not. A graphical representation of a planar graph without edge intersections is called a *plane drawing*. Figure 3-1A shows a planar graph and Figure 3-1B shows a plane drawing of the same graph.

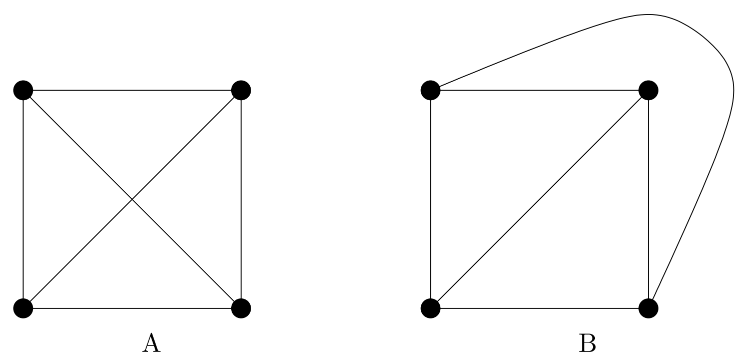


**Figure 3-1.** The $K_4$, a planar graph. The graphs A and B are the same, but only B is a plane drawing.

The *faces* of a plane drawing of a planar graph $G$ are the regions of the plane that are separated from each other by the edges of $G$. That is, it is not possible to draw a continuous curve from a point within a region to a point outside of this region without crossing an edge. Figure 3-2 shows a plane graph $K_4$, which has four faces. One of the

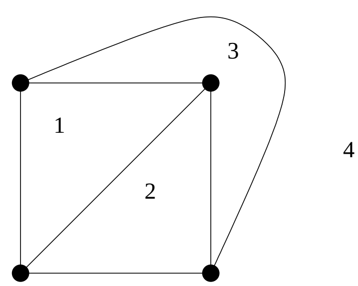


**Figure 3-2.** A plane drawing of $K_4$ showing its four faces (1-4). Notice that Face 4 is the unbounded face lying outside the graph.

faces of any graph is the *unbounded face* that covers the majority of the plane. A graph with no edges has only a single face, the unbounded face.

**Theorem 3-1** (Euler). *Let $G$ be a plane drawing of a connected planar graph. Let $n$, $m$, and $f$ denote the number of vertices, edges, and faces. Then*

$$n - m + f = 2. \tag{3-1}$$

*Proof:* The proposition will be proved by induction on m. If $m = 0$, then $G$ has one vertex, no edges, and one face: $n = 1$, $m = 0$, $f = 1$. So Equation (3-1) holds for the base case. Suppose Equation (3-1) is true for any graph with at most $m - 1$ edges. If $G$ is a tree, then $m = n - 1$, and $f = 1$; so Equation (3-1) holds. If $G$ is not a tree, let $e$ be an edge of some cycle of $G$. Then $G$ with the edge $e$ removed is a connected plane graph with $n$ vertices, $m - 1$ edges, and $f - 1$ faces. So $n - (m - 1) + (f - 1) = 2$ by the induction hypothesis. Hence $n - m + f = 2$. □

**Corollary 3-2.** *Let $G$ be a plane drawing of a connected planar graph. Let $n$ and $m$ denote the number of vertices and edges and let $m \geq 3$. Then*

$$m \leq 3n - 6. \tag{3-2}$$

*Proof:* Let $f$ denote the number of faces. Every face is surrounded by at least 3 edges. Every edge has one and only one face on either of its two sides (note that the same face may be on both sides of an edge, such as in the case of a tree). We conclude that $3f \leq 2m$. Plugging the inequality into Euler's formula, we get Equation (3-2). □

**Corollary 3-3.** *Any planar graph contains a vertex of degree at most 5.*

*Proof:* Assume every vertex has degree 6 or more. Then $6n \leq 2m$ since every vertex is the end of at least six edges. So $3n \leq m$. But then $3n \leq 3n - 6$ by Equation (3-2), a contradiction for all $n$. Therefore there must be at least a single vertex with degree at most 5. □

Let $G$ be a plane drawing of a graph. Its *dual* graph $G^*$ is constructed by placing exactly one vertex in every face of the graph $G$, and then constructing edges such that each edge crosses one of the original edges, and connects the vertices on either side of that edge. Figure 3-3 shows a sample graph $G$ and its dual $G^*$. Duals will often have loops.

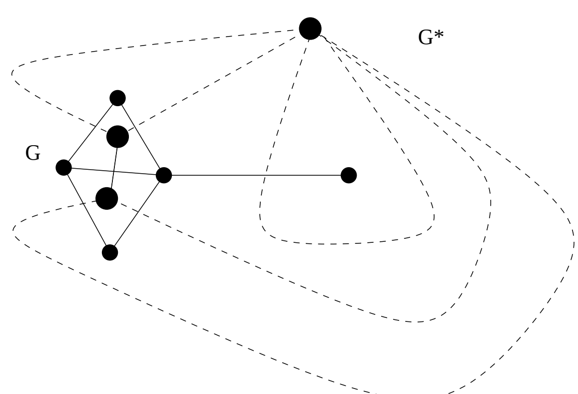


**Figure 3-3.** A graph and its dual. The dual is indicated by larger vertices and dashed edges.

It can be shown that the dual of the dual is the original graph; see [**5**, p. 76]. The notion of duality is very important because it allows us to relate the coloring of faces to the coloring of vertices. Also, if the edges of $G$ are in one-to-one correspondence with the edges in $G^*$, then the dual is called an *abstract dual.* It can be shown that a graph is planar if and only if it has an abstract dual; see [**5**, p. 76].

**4. Coloring a Graph.** A graph may be colored in several ways. We may color either the vertices, the edges, or the faces. We will concentrate on coloring the vertices. The rules are as follows: each vertex must be colored; and no two vertices connected by an edge may be the same color.

A *map* is a planar graph with no "bridges"; that is, removing any single edge will not disconnect the graph. The Four Color Theorem asserts that any map can have its faces colored with at most four different colors such that no two faces that share an edge are the same color. Clearly, the Four Color Theorem can be expressed in terms of the dual graph as follows: the dual's vertices can be colored with at most four colors such that no two vertices connected by an edge are colored the same color. Notice that no map's dual will have a loop or more than one edge between any two vertices; so our earlier restriction on planar graphs is satisfied. Moreover, a planar graph's dual is also a planar graph, see [**5**, p. 73]. Thus, to prove the Four Color Theorem, it is equivalent to prove that no planar graph requires more than four colors to color its vertices. For the sake of simplicity, from now on, we will refer to a graph as being *k-colorable* if its vertices can be colored with $k$ colors.

Let us take a small detour, and prove that $K_5$ is not planar.

**Proposition 4-1.** *The graph $K_5$ is not planar.*

*Proof:* Suppose $K_5$ is planar. Figure 4-1 shows a drawing of $K_5$. Note that $K_5$ has the cycle

$$a \to b \to c \to d \to e \to a.$$

That cycle is of length 5, so let us refer to it as a pentagon. Any plane drawing of $K_5$ must of course contain this pentagon. Since $K_5$ is planar, none of the edges of the pentagon intersect. Each of the edges $ac$, $ad$, $be$, $bd$, and $ec$ must each lie either inside the pentagon or outside the pentagon.

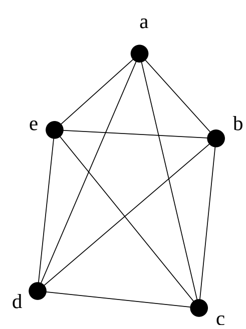


**Figure 4-1.** A graphical representation of $K_5$.

Let us consider the case that $ac$ lies inside the pentagon. In order for the edges $be$ and $bd$ to not cross $ac$, they must lie outside the pentagon (see Figure 4-2). The two edges that are left, $ec$ and $ad$, cannot both lie outside the pentagon because then they would cross. Similarly, they also cannot both lie inside the pentagon. So one edge must lie inside the pentagon and one outside. The edge $ec$ cannot lie on the outside because if it did it would cross $bd$. The edge $ad$ cannot lie on the outside because if it did it would cross $be$. We have a contradiction. Therefore $K_5$ is not planar. □

In 1930, Kazimierz Kuratowski proved that no graph is planar if it contains either $K_5$ or $K_{3,3}$ (see Figure 2-4 for representations of both graphs) or is contractible to either

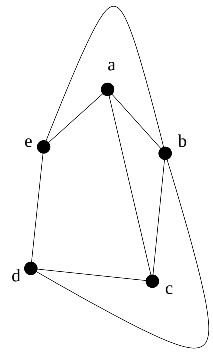


**Figure 4-2.** The graph $K_5$ without two edges: $ec$ and $ad$.

$K_5$ or $K_{3,3}$. *Contracting* means taking an edge and removing it, making its two vertices into a single vertex with all the edges of the two old vertices connected to the new one. In fact, later, Kuratowski proved a graph is planar if it does not contain a subgraph that is $K_5$ or $K_{3,3}$, and is not contractible to $K_5$ or $K_{3,3}$. Therefore, the Four Color Theorem applies to all graphs that do not contain $K_5$ or $K_{3,3}$, and are not contractible to $K_5$ or $K_{3,3}$. The proof is too long to include in this paper; see [**3**, p. 95].

**Theorem 4-2.** *Let $G$ be a planar graph with largest vertex degree $D$. Then $G$ is $(D+1)$-colorable.*

*Proof:* We proceed by induction on the number of vertices $n$. If $n = 1$, then $D = 0$, and the graph is 1-colorable; so the theorem holds for the base case. Assume the theorem is true for a graph with $n-1$ vertices. Remove a vertex $v$ and all the edges that are connected to it. The graph that remains has $n-1$ vertices and is $(D+1)$-colorable if $D$ is the largest degree of a vertex. Return the vertex and color it a different color from vertices that are connected to it. There are at most $D$ such vertices. Hence we need at most $D+1$ colors. □

The next step is to prove that all planar graphs are 6-colorable, and therefore all maps are 6-colorable.

**Theorem 4-3.** *All maps are 6-colorable.*

*Proof:* In fact, we show that all planar graphs are 6-colorable. We proceed by induction on the number of vertices $n$. Let $G$ be a planar graph with $n$ vertices. If $n \leq 5$, then $G$ is 6-colorable since there are less than six vertices to color. Assume all graphs with $n-1$ vertices are 6-colorable. According to Corollary 3-3, $G$ contains a vertex $v$ of degree at most five. Remove $v$ and all the edges connected to it. The graph left has $n-1$ vertices, and is therefore 6-colorable. Return the vertex $v$ and all the edges that were previously present (at most five of them) to the graph and color $v$ a different color from vertices connected to it. Thus there exists a coloring of $G$ with at most six colors. □

We can even go further and show that all planar graphs are 5-colorable with fairly simple methods.

**Theorem 4-4.** *All maps are 5-colorable.*

*Proof:* In fact, we show that all planar graphs are 5-colorable. We proceed by induction on the number of vertices $n$. Let $G$ be a planar graph with $n$ vertices. If $n \leq 5$, then $G$ is 5-colorable because there are no more than five vertices to color.

Assume all planar graphs with $n-1$ vertices are 5-colorable. According to Corollary 3-3, $G$ contains a vertex $v$ of degree at most 5. Remove $v$ and all the edges connected to it. The graph left has $n-1$ vertices and is therefore 5-colorable. Return the vertex $v$ and all the edges that were previously present. Our goal is to color the vertex $v$ with one of the five colors available to us. If the degree of $v$ is less than 5, then $v$ can be colored one of the five available colors different from the colors of the vertices to which $v$ is connected. Suppose the degree of $v$ is 5, and say the five edges of $v$ lead to vertices $v1$, $v2$, $v3$, $v4$, and $v5$, which are arranged around $v$ in a clockwise order (see Figure 4-3).

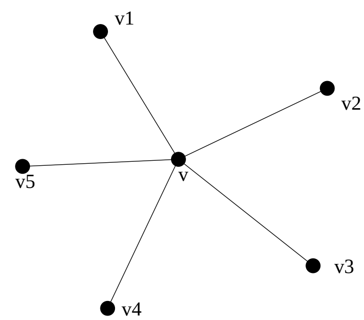


**Figure 4-3.** The clockwise arrangement of vertices $v1$ through $v5$ around $v$.

If there exists an edge connecting each pair of vertices of the set of vertices $v1$ through $v5$, then $G$ contains $K_5$ and is not planar, a contradiction. Hence at least one edge must be missing from the connections of $v1$ through $v5$; that is, at least two of the vertices, say $v1$ and $v3$, are not connected by an edge. Let us contract the two edges from $v$ to the two unconnected vertices. The resulting graph has fewer than $n$ vertices, and therefore is 5-colorable. Return the two edges and give $v1$ and $v3$ the same color, the color originally assigned to the vertex that was the result of the contraction of the edges. Now, color the vertex $v$ one of the five colors available, but different from the at most four colors used on vertices $v1$ through $v5$. Thus all planar graphs are 5-colorable. □

## References

[**1**] Allaire, F., *Another Proof of the Four Colour Theorem — Part I*, in "Proc. Of the 7th Manitoba Conference on Numerical Mathematics and Computing," 1977.

[**2**] Appel, K., and Haken, W., *Every Planar Map is Four Colorable*, A.M.S. Contemporary Mathematics **98** (1989), 236–240.

[**3**] Murty, U., "Graph Theory with Applications," American Elsevier, 1979.

[**4**] Robertson, N., Sanders, D., Seymour, P., and Thomas, R., *A New Proof of the Four-Colour Theorem*, Electronic Research Announcements of the American Mathematical Society **2**(1) (1996), 84–96.

[**5**] Wilson, R., "Introduction to Graph Theory," Fourth Edition, Addison Wesley Longman Limited, England, 1996.