Homework 0: Math Review

CS 490A, UMass Amherst, Fall 2021

Note

Wikipedia can be a useful resource for basic probability, linear algebra, and calculus.

Please write your answers on paper and scan, or in LaTeX as you like, and upload the PDF to Gradescope before class on Thursday Sept. 9.

1 Domain of a joint distribution

1.1

A and B are discrete random variables. A can take on one of 3 possible values. B can take on one of 7 possible values. (In other words, these are the sizes of domain(A) and domain(B).) How many possible outcomes does the joint distribution $P(A, B)$ define probabilities for?

1.2

Say we have a sequence of n binary random variables $A_1, A_2, \ldots A_n$. How many possible outcomes does the joint distribution $P(A_1, A_2, \ldots A_n)$ define probabilities for?

2 Probabilistic Definitions and Independence

Say we have three random variables A, B and C. Note that we're using standard probability theory notation where $P(A, B)$ means the joint probability of both A and B occurring.

2.1

Which of the following statements are always true?

1. $P(A|B) = P(B|A)$
2. $P(A, B) = P(A|B)P(B)$
3. $P(A, B) = P(A)P(B)$
4. $P(A|B) = P(A)$
5. $P(A, B, C) = P(A)P(C)$
6. $P(A, B, C) = P(A)P(B)P(C)$
7. $P(A, B, C) = P(A)P(B|A)P(C|A, B)$

8. $P(A) = \sum_{b \in \text{domain}(B)} P(A, B = b)$
9. $P(A) = \sum_{b \in \text{domain}(B)} P(A|B = b)P(B = b)$
10. $\log(P(A)P(B)) = \log P(A) + \log P(B)$

2.2

Now assume that A, B, and C are all independent of each other. Which of the above statements are now true?

3 Logarithms

3.1 Log-probs

Let p be a probability, so it is bounded to $[0, 1]$. This is equivalent to saying $p \in [0, 1]$, or that $0 \leq p \leq 1$, or that p is between 0 and 1, inclusive.

3.1.1

For what values of p is $\log(p)$ defined?

3.1.2

What is the range of possible values for $\log(p)$?

3.2 Prob ratios

Let p and q both be probabilities.

3.2.1

For what values of p and q is p/q defined?

3.2.2

What is the range of possible values for p/q?

3.3 Log prob ratios

What is the range of possible values for $\log(p/q)$?

4 Linear algebra

$\boldsymbol{x}$ is a 10-d real-valued vector (i.e., $\boldsymbol{x} \in \mathbb{R}^{10}$). $\boldsymbol{y}$ is another vector of the same dimensionality ($\boldsymbol{y} \in \mathbb{R}^{10}$). $\mathbf{W}_1$ is a real-valued matrix of dimensionality 10×10, and $\mathbf{W}_2$ is a real-valued matrix of dimensionality 20×10.

Answer the following questions. Feel free to look at online resources such as Wikipedia for help, and/or additionally test out your answers programmatically using libraries such as numpy.

1. What is the dimensionality of the element-wise product $\boldsymbol{x} * \boldsymbol{y}$?
2. What is the dimensionality of the dot product of $\boldsymbol{x}$ and $\boldsymbol{y}$? (Other equivalent notation: $\boldsymbol{x} \cdot \boldsymbol{y}$, or $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$, or $\boldsymbol{x}^\intercal \boldsymbol{y}$, or $\boldsymbol{x}' \boldsymbol{y}$. The last two are pronounced "x transpose y," via the interpretation that a vector is a one-column matrix which can be transposed.)
3. Is $\mathbf{W}_1 \boldsymbol{x}$ a proper expression, and if so, what is its dimensionality?
4. Is $\mathbf{W}_1 \mathbf{W}_2 \boldsymbol{x}$ a proper expression, and if so, what is its dimensionality?
5. Is $\mathbf{W}_2 \mathbf{W}_1 \boldsymbol{x}$ a proper expression, and if so, what is its dimensionality?
6. Assume the magnitude (a.k.a. L2 norm) of $\boldsymbol{x}$ is 1 (i.e., $\|\boldsymbol{x}\| = \sqrt{\sum_i x_i^2} = 1$.) What is $\boldsymbol{x} \cdot \boldsymbol{x}$?