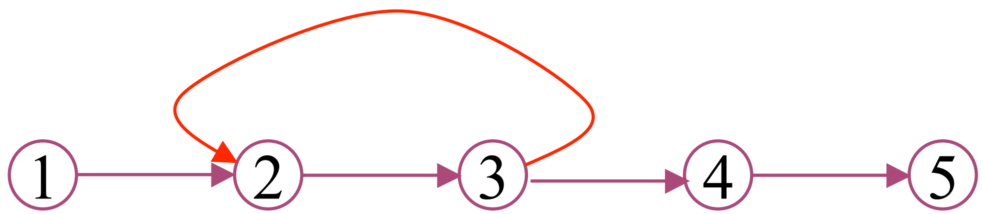
Chapter 5: Monte Carlo Methods

- Monte Carlo methods learn from *complete* sample returns
  - Only defined for episodic tasks
- Monte Carlo methods learn directly from experience
  - *On-line:* No model necessary and still attains optimality
  - *Simulated:* No need for a *full* model

Monte Carlo Policy Evaluation

- *Goal:* learn $V^{\pi}(s)$
- *Given:* some number of episodes under π which contain s
- *Idea:* Average returns observed after visits to s



- *Every-Visit MC:* average returns for *every* time s is visited in an episode
- *First-visit MC:* average returns only for *first* time s is visited in an episode
- Both converge asymptotically

First-visit Monte Carlo policy evaluation

Initialize:

$\pi \leftarrow$ policy to be evaluated

$V \leftarrow$ an arbitrary state-value function

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Repeat forever:

(a) Generate an episode using π

(b) For each state s appearing in the episode:

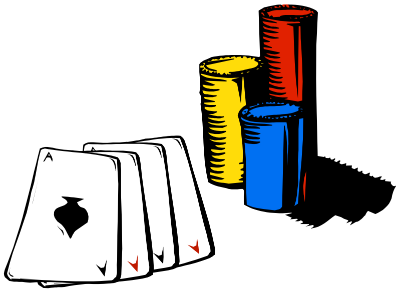
$R \leftarrow$ return following the first occurrence of s

Append R to $Returns(s)$

$V(s) \leftarrow \text{average}(Returns(s))$

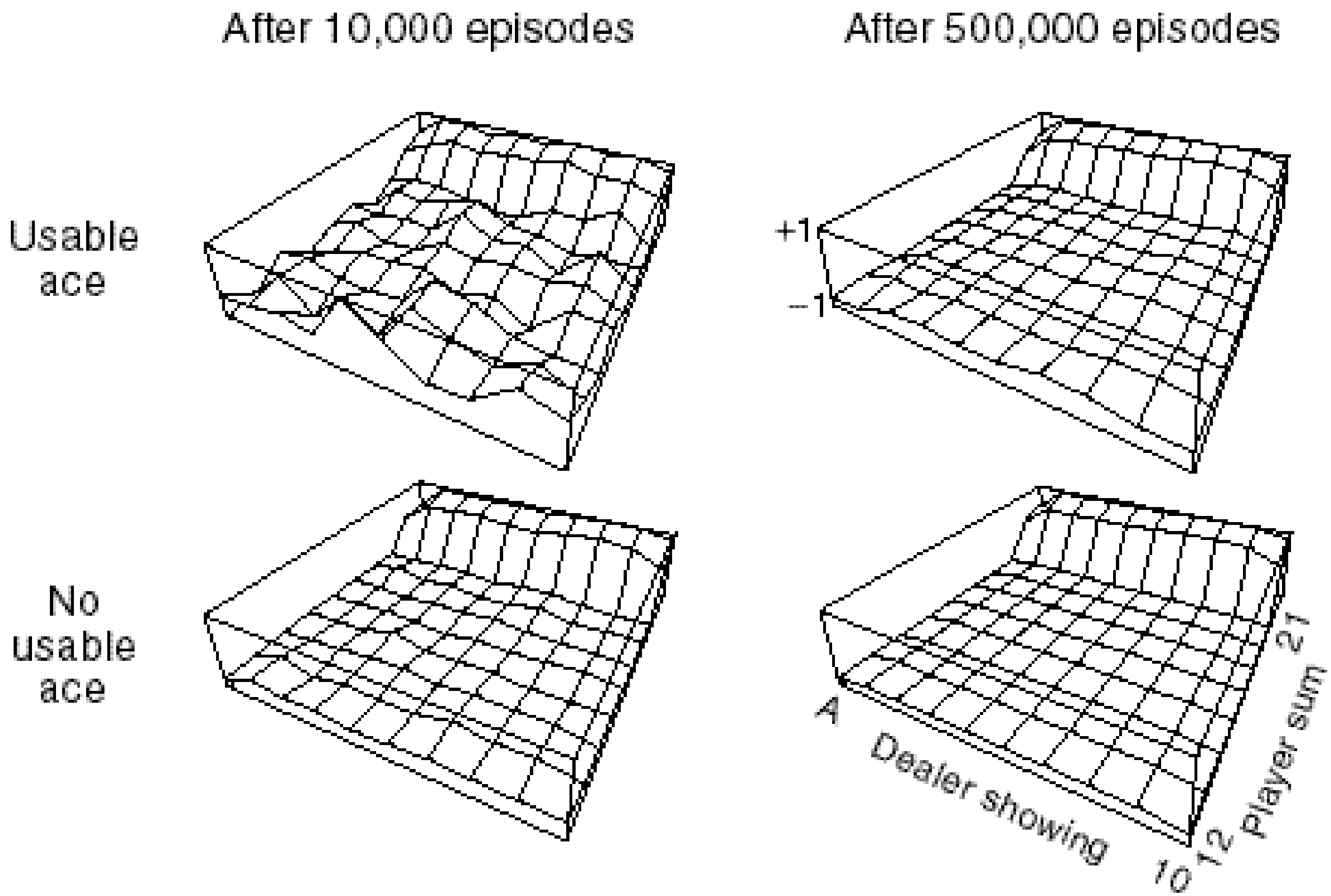
Blackjack example

- *Object:* Have your card sum be greater than the dealers without exceeding 21.
- *States* (200 of them):
  - current sum (12-21)
  - dealer's showing card (ace-10)
  - do I have a useable ace?



- *Reward:* +1 for winning, 0 for a draw, –1 for losing
- *Actions:* stick (stop receiving cards), hit (receive another card)
- *Policy:* Stick if my sum is 20 or 21, else hit

Blackjack value functions

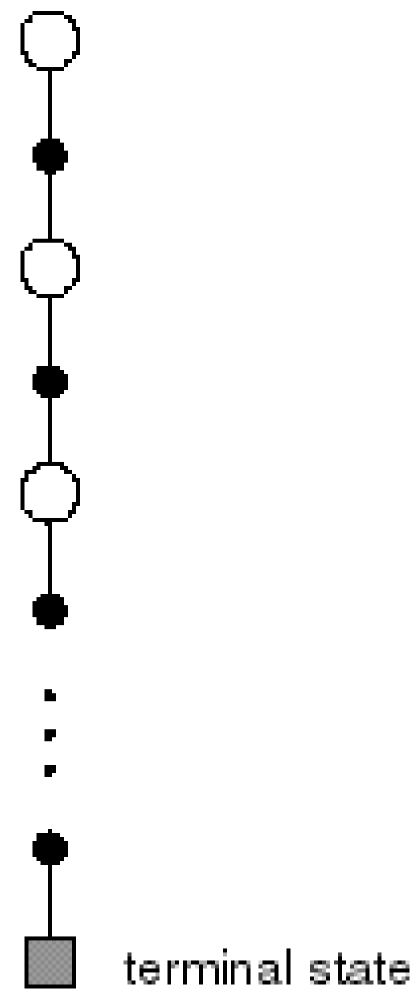


Backup diagram for Monte Carlo

- Entire episode included
- Only one choice at each state (unlike DP)

- MC does not bootstrap

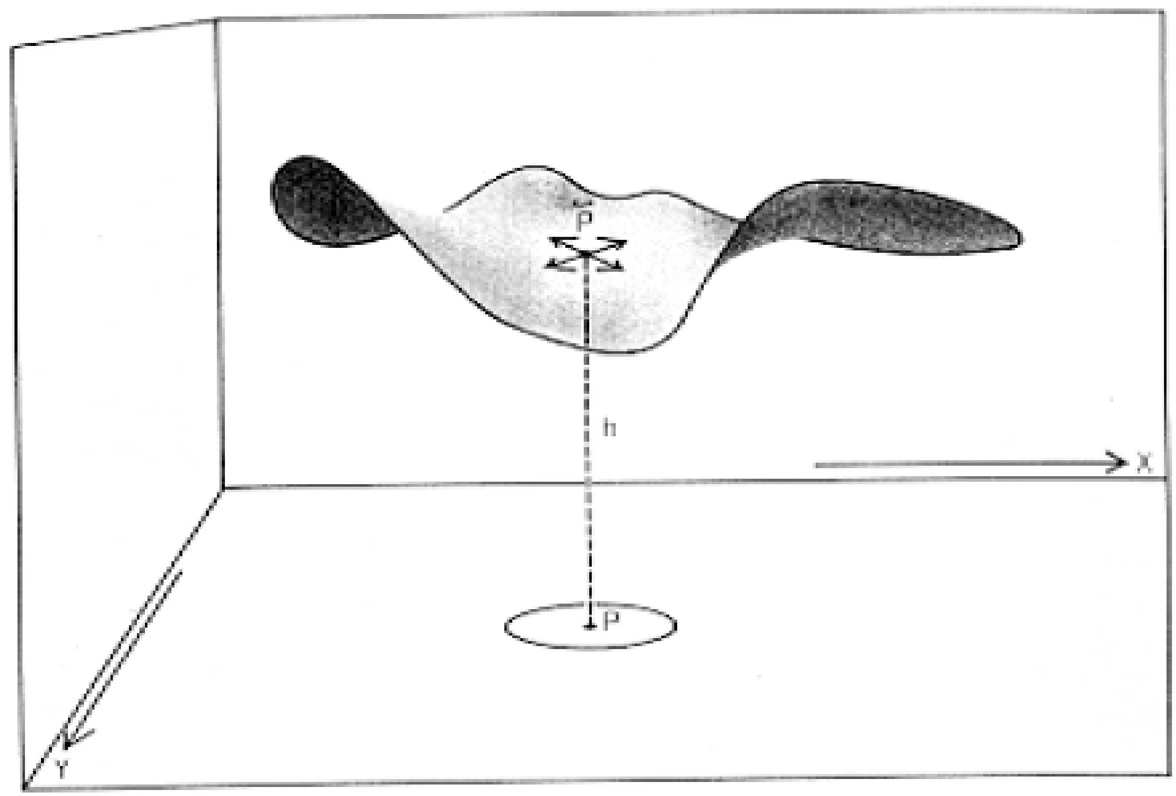
- Time required to estimate one state does not depend on the total number of states



The Power of Monte Carlo

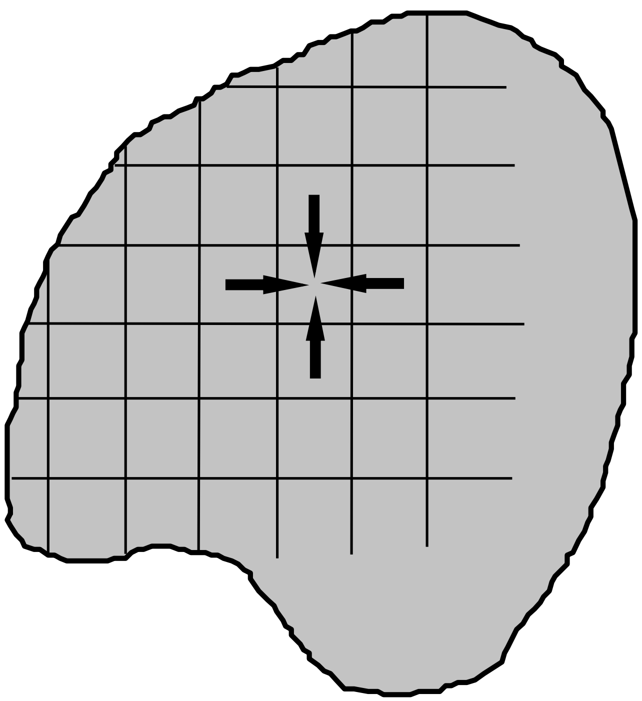
e.g., Elastic Membrane (Dirichlet Problem)

How do we compute the shape of the membrane or bubble?

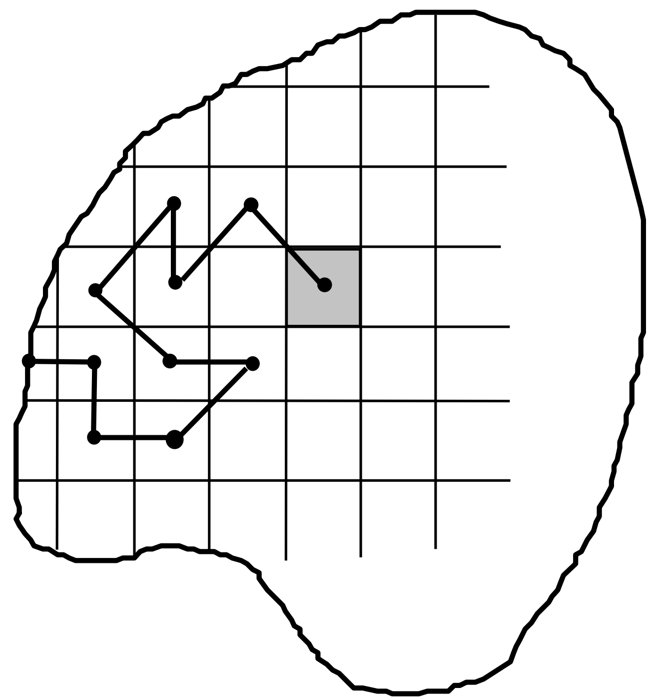


Two Approaches

Relaxation



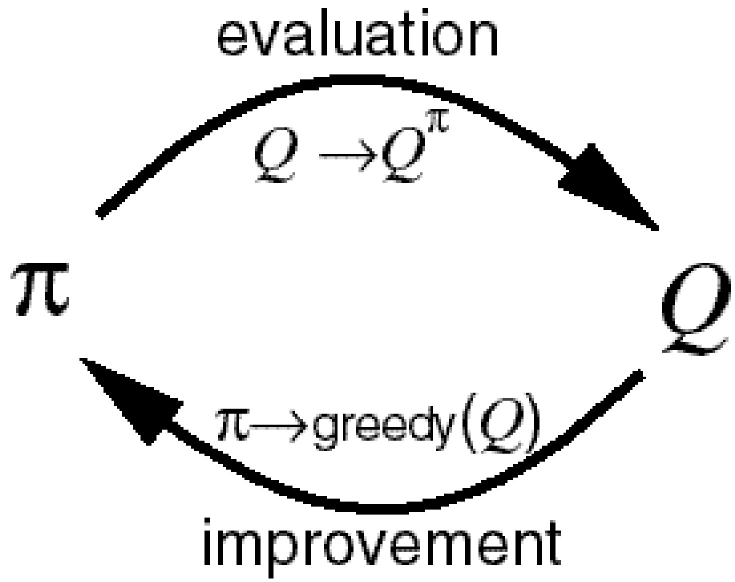
Kakutani's algorithm, 1945



Monte Carlo Estimation of Action Values (Q)

- Monte Carlo is most useful when a model is not available
  - We want to learn Q^*
- $Q^\pi(s,a)$ - average return starting from state s and action a following π
- Also converges asymptotically *if* every state-action pair is visited
- *Exploring starts:* Every state-action pair has a non-zero probability of being the starting pair

Monte Carlo Control



- MC policy iteration: Policy evaluation using MC methods followed by policy improvement
- Policy improvement step: greedify with respect to value (or action-value) function

Convergence of MC Control

- Greedified policy meets conditions for policy improvement:

$$
\begin{aligned}
Q^{\pi_k}(s, \pi_{k+1}(s)) &= Q^{\pi_k}(s, \arg\max_a Q^{\pi_k}(s,a)) \\
&= \max_a Q^{\pi_k}(s,a) \\
&\geq Q^{\pi_k}(s, \pi_k(s)) \\
&= V^{\pi_k}(s)
\end{aligned}
$$

- And thus must be $\geq \pi_k$ by the policy improvement theorem
- This assumes exploring starts and infinite number of episodes for MC policy evaluation
- To solve the latter:
  - update only to a given level of performance
  - alternate between evaluation and improvement per episode

Monte Carlo Exploring Starts

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \leftarrow$ arbitrary
  $\pi(s) \leftarrow$ arbitrary
  $Returns(s,a) \leftarrow$ empty list

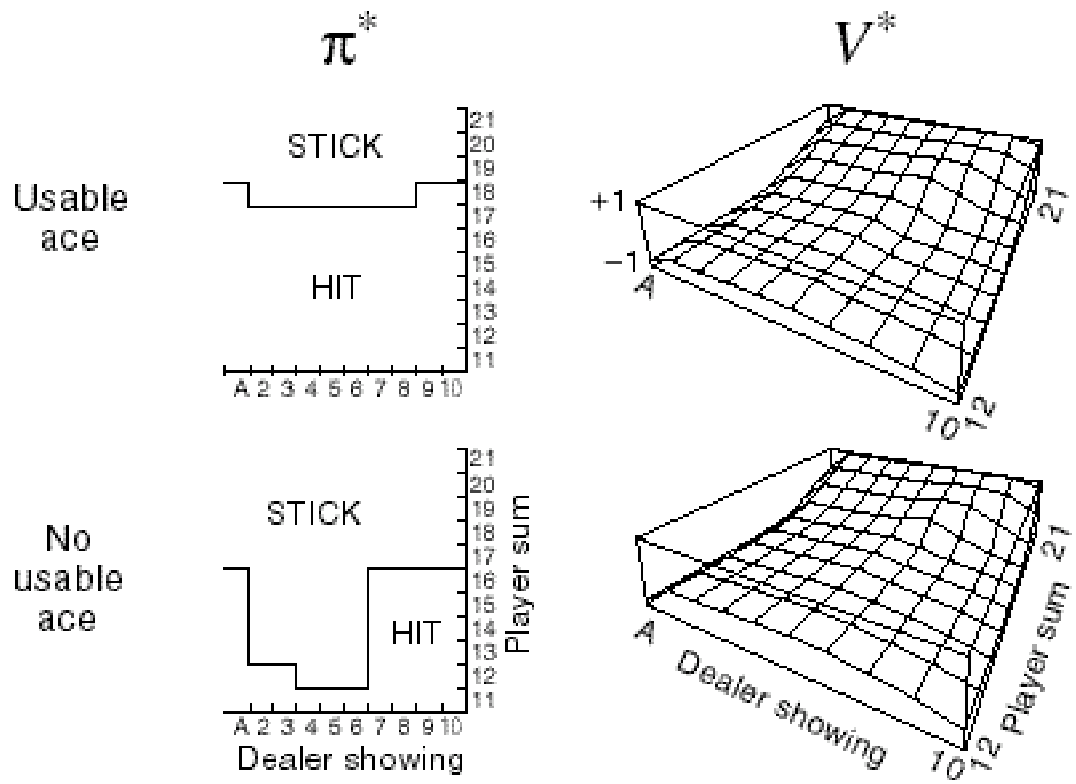
Repeat forever:
  (a) Generate an episode using exploring starts and π
  (b) For each pair s, a appearing in the episode:
    $R \leftarrow$ return following the first occurrence of s, a
    Append R to $Returns(s,a)$
    $Q(s,a) \leftarrow \text{average}(Returns(s,a))$
  (c) For each s in the episode:
    $\pi(s) \leftarrow \arg\max_a Q(s,a)$

Fixed point is optimal policy π^*

Proof is open question

Blackjack example continued

- Exploring starts
- Initial policy as described before



On-policy Monte Carlo Control

- *On-policy:* learn about policy currently executing
- How do we get rid of exploring starts?
  - Need *soft* policies: $\pi(s,a) > 0$ for all s and a
  - e.g. ε-soft policy:

$$\frac{\varepsilon}{|A(s)|} \qquad\qquad 1 - \varepsilon + \frac{\varepsilon}{|A(s)|}$$

non-max greedy

- Similar to GPI: move policy *towards* greedy policy (i.e. ε-soft)
- Converges to best ε-soft policy

On-policy MC Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \leftarrow$ arbitrary
  $Returns(s,a) \leftarrow$ empty list
  $\pi \leftarrow$ an arbitrary ε-soft policy

Repeat forever:
  (a) Generate an episode using π
  (b) For each pair s, a appearing in the episode:
    $R \leftarrow$ return following the first occurrence of s, a
    Append R to $Returns(s,a)$
    $Q(s,a) \leftarrow \text{average}(Returns(s,a))$
  (c) For each s in the episode:
    $a^* \leftarrow \arg\max_a Q(s,a)$
    For all $a \in \mathcal{A}(s)$:

$$\pi(s,a) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(s)| & \text{if } a = a^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq a^* \end{cases}$$

Off-policy Monte Carlo control

- Behavior policy generates behavior in environment
- Estimation policy is policy being learned about
- Average returns from behavior policy by probability their probabilities in the estimation policy

Learning about π while following π'

Suppose we have n_s returns, $R_i(s)$, from state s, each with probability $p_i(s)$ of being generated by π and probability $p_i'(s)$ of being generated by π'. Then we can estimate

$$V^\pi(s) \approx \frac{\sum_{i=1}^{n_s} \frac{p_i(s)}{p_i'(s)} R_i(s)}{\sum_{i=1}^{n_s} \frac{p_i(s)}{p_i'(s)}}$$

which depends on the environmental probabilities $p_i(s)$ and $p_i'(s)$. However,

$$p_i(s_t) = \prod_{k=t}^{T_i(s)-1} \pi(s_k, a_k) \mathcal{P}_{s_k s_{k+1}}^{a_k}$$

and

$$\frac{p_i(s_t)}{p_i'(s_t)} = \frac{\prod_{k=t}^{T_i(s)-1} \pi(s_k, a_k) \mathcal{P}_{s_k s_{k+1}}^{a_k}}{\prod_{k=t}^{T_i(s)-1} \pi'(s_k, a_k) \mathcal{P}_{s_k s_{k+1}}^{a_k}} = \prod_{k=t}^{T_i(s)-1} \frac{\pi(s_k, a_k)}{\pi'(s_k, a_k)}.$$

Thus the weight needed, $p_i(s)/p_i'(s)$, depends only on the two policies and not at all on the environmental dynamics.

Off-policy MC control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

$Q(s,a) \leftarrow$ arbitrary

$N(s,a) \leftarrow 0$; Numerator and

$D(s,a) \leftarrow 0$; Denominator of $Q(s,a)$

$\pi \leftarrow$ an arbitrary deterministic policy

Repeat forever:

(a) Select a policy π' and use it to generate an episode:

$$s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T, s_T$$

(b) $\tau \leftarrow$ latest time at which $a_\tau \neq \pi(s_\tau)$

(c) For each pair s, a appearing in the episode after τ:

$t \leftarrow$ the time of first occurrence (after τ) of s, a

$w \leftarrow \prod_{k=t+1}^{T-1} \frac{1}{\pi'(s_k, a_k)}$

$N(s,a) \leftarrow N(s,a) + wR_t$

$D(s,a) \leftarrow D(s,a) + w$

$Q(s,a) \leftarrow \frac{N(s,a)}{D(s,a)}$

(d) For each $s \in \mathcal{S}$:

$\pi(s) \leftarrow \arg\max_a Q(s,a)$

Incremental Implementation

- MC can be implemented incrementally
  - saves memory
- Compute the weighted average of each return

$$V_n = \frac{\sum_{k=1}^{n} w_k R_k}{\sum_{k=1}^{n} w_k}$$

non-incremental

$$V_{n+1} = V_n + \frac{w_{n+1}}{W_{n+1}}\left[R_{n+1} - V_n\right]$$

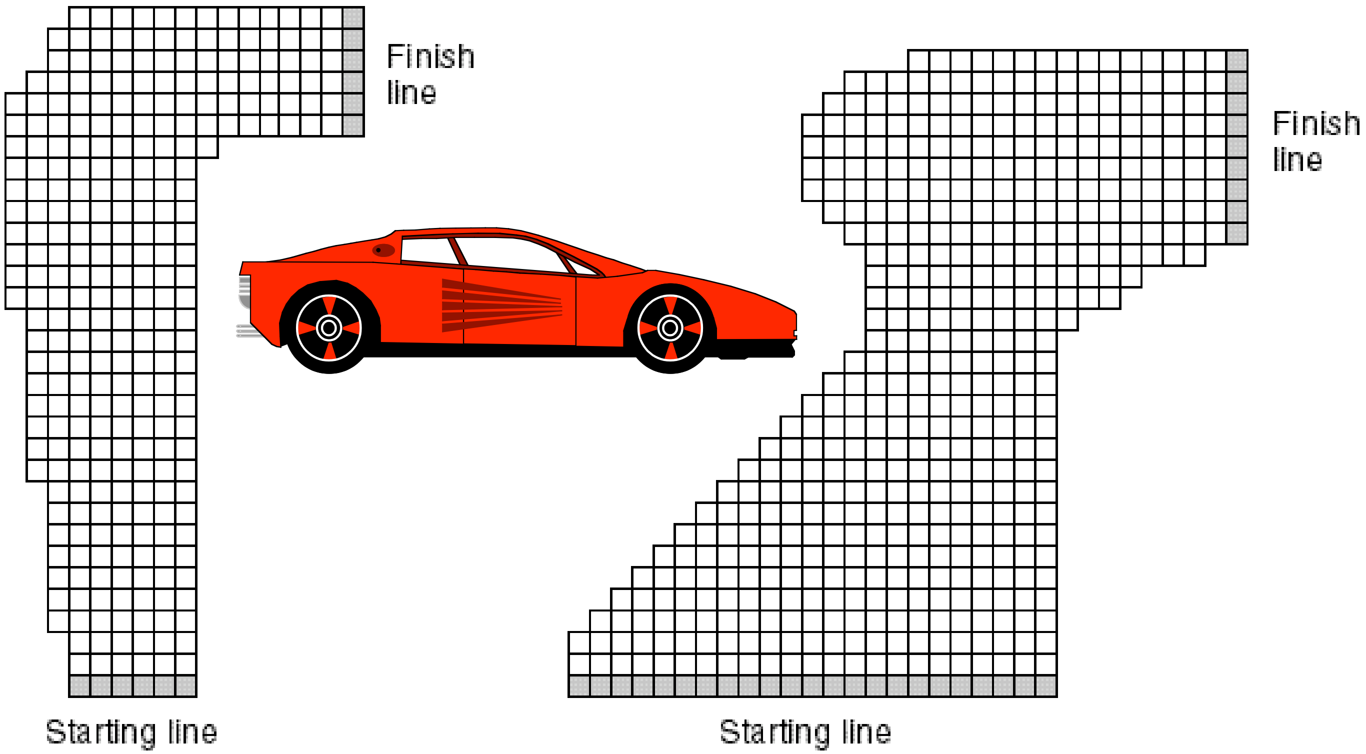
$$W_{n+1} = W_n + w_{n+1}$$

$$V_0 = W_0 = 0$$

incremental equivalent

Racetrack Exercise

- *States:* grid squares, velocity horizontal and vertical
- *Rewards:* -1 on track, -5 off track
- *Actions:* +1, -1, 0 to velocity
- 0 < Velocity < 5
- Stochastic: 50% of the time it moves 1 extra square up or right



Summary

- MC has several advantages over DP:
  - Can learn directly from interaction with environment
  - No need for full models
  - No need to learn about ALL states
  - Less harm by Markovian violations (later in book)
- MC methods provide an alternate policy evaluation process
- One issue to watch for: maintaining sufficient exploration
  - exploring starts, soft policies
- No bootstrapping (as opposed to DP)