NAME: ____________________

SPIRE ID: ____________________

INFO 150
A Mathematical Foundation for Informatics
Second Midterm Exam Fall 2025

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 100 total points. Probable scale is somewhere around A=90, C=60, but will be determined after I grade the exam.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/15
3	/15
4	/15
5	/20
6	/20
Total	/100

Question 1 (15): Briefly identify and distinguish the meaning of each of these terms or concepts (3 points each):

- (a) the **inductive hypothesis** of an induction and its **inductive goal**

- (b) the **intersection** of two sets A and B and the **union** of those two sets

- (c) an **antisymmetric** binary relation and a binary relation that is **not symmetric**

- (d) a **function** from X to Y and an **onto** function from X to Y

- (e) the **rule of sums** and the **rule of sums with overlap** in counting two finite sets A and B

Question 2 (15): Here we define a function f from positive integers to positive integers, using the following recursive definition. We define $f(1)$ to be 1. If n is even, then $f(n)$ is defined to be $f(n/2)$. If n is odd and $n > 1$, then $f(n)$ is defined to be $f(n-2)+2$. Your goal is to prove, for all positive integers n, that $f(n)$ is odd.

- (a, 3) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

- (b, 3) State and prove the **base case** (or base cases) for your induction.

- (c, 3) State the **inductive hypothesis** and **inductive goal** for your inductive step.

- (d, 6) Prove your inductive step, completing the proof.

Question 3 (15): Let D be a set of dogs, including a set L of Labradors, a set R of retrievers, and a set T of terriers.

- (a, 3) We are told that "Every Labrador is a retriever, but no retriever is a terrier." Express this statement using subsets, using set operations and the relations $\subseteq$ and/or $=$ on the given sets.

- (b, 3) Draw a Venn diagram showing the relationship of all four sets D, L, R, and T as given at the beginning of the question and in part (a).

- (c, 3) Here is another statement: "Every terrier is not a Labrador". What would be a *counterexample* for this statement?

- (d, 6) Write a letter to the Reader that should convince them, given the statement in part (a), that they should not look for counterexamples to the statement in part (c).

Question 4 (15): Consider two functions from $\mathbb{Z}$ (the integers, including negative numbers) to $\mathbb{Z}$. For any integer n, g is defined such that $g(n) = 2n - 3$ and h is defined such that $h(n) = n - 4$.

- (a, 3) What functions are the **compositions** $g \circ h$ and $h \circ g$?

- (b, 4) Are either or both of the functions g and h **one-to-one functions** (injections)? Justify your answers.

- (c, 4) Are either of both of the functions g and h **onto functions** (surjections)? Justify your answers.

- (d, 4) Do either or both of the functions g and h have **inverse functions**? Give the inverses, if any, where they exist.

Question 5 (20): In this problem, we consider randomly choosing a three-letter word, where each letter is chosen from the set $\{A, B, C, D, E\}$. We assume that in each of the three positions, each of the five letters are equally likely to be chosen. We also assume that the choices of the three letters are **independent**.

1. (a, 3) In how many ways could the three letters be chosen?

2. (b, 2) What is the probability that the word chosen is exactly "EAD"?

3. (c, 3) In how many different ways could the word be chosen with three different letters? (For example, "EAD" would be one of these words, but "EAE" would not be.)

4. (d, 2) What is the probability that the event of part (c) happens?

5. (e, 3) In how many ways could the word be chosen such that the letters are all different, *and* that the letters occur in order? (For example, "ADE" would be one of these words, but "AED" would not be.)

6. (f, 2) What is the probability that the event of part (e) happens?

7. (g, 3) In how many ways could the word be chosen such the letters come in order, whether or not the letters are different? (For example, "AAD", "BCC", "EEE" and "ABE" would all be among these strings, but "CBC" or "BBA" would not be.)

8. (h, 2) What is the probability that the event of part (g) happens?

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Let $P(n)$ be a predicate on the positive integers. If $P(1)$ is true, and for all n with $n > 2$ we know that $P(n-2) \rightarrow P(n)$, we *cannot* be confident that $P(n)$ is true for all positive integers.

- (b) If $P(m-1) \rightarrow P(m)$ is true for all positive integers m, then we know that $P(n)$ is true for all positive integers n.

- (c) Let $C = \{4k+1 : k \in \mathbb{Z}\}$ and $D = \{4k+3 : k \in \mathbb{Z}\}$. Whatever sets A and B are, we can be sure that $\{A, B, C, D\}$ is not a partition of $\mathbb{Z}$.

- (d) The set $\{m^2 : m \in \mathbb{Z} \wedge (-4 \leq m \leq 4)\}$ has fewer than nine elements.

- (e) If a binary relation on a nonempty set A has no pairs in it at all, it is symmetric and transitive, but is not reflexive.

- (f) Let A and B be any two nonempty sets, and let R be a relation from A to B. If there is an element $x \in A$ such that there is no pair (x, y) in R for any y in B, then R is not a function.

- (g) If A, B, and C are any three finite sets, the size of $A \cup B \cup C$ (called $|A \cup B \cup C|$ is equal to $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

- (h) There are exactly six binary strings of length 3 that do not have all three of their bits the same.

- (i) Suppose I flip four fair coins, with each toss being independent. Then the probability that there are two heads and two tails is greater than or equal to 1/2.

- (j) If X and Y are any two events over the same probability space, then $Prob(X \cup Y) = Prob(X) + Prob(Y)$.