Turing Machines

$$M = (Q, \Sigma, \delta, s)$$

Q: finite set of states; $s \in Q$

Σ: finite set of symbols; $\triangleright, \sqcup \in \Sigma$

δ: $Q \times \Sigma \ \rightarrow \ (Q \cup \{h\}) \times \Sigma \times \{\leftarrow, \rightarrow, -\}$

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$

TM's are exactly like DFA's, *except*

- They may move **either way** on their tape
- They may **change** tape contents
- They have **unlimited extra memory** on the right end of the tape

Giving a DFA some but not all of these capabilities gives some intermediate models of computation:

- The **two-way DFA** can still only decide regular languages, though perhaps with many fewer states than the corresponding ordinary DFA. Proving this is a good exercise in the use of the Myhill-Nerode Theorem.
- The **linear-bounded automaton** can change its tape, but must stay within the bounds of the original input. It recognizes the class we'll later call **DSPACE**(n) and has a corresponding grammar definition.

Example TM

mvRt.tm	s	q	q_0	q_1
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$		
1	$s, 1, \rightarrow$	$q_1, \sqcup, \rightarrow$		
$\sqcup$	$q, \sqcup, \leftarrow$		$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$		
comment	find $\sqcup$	memorize & erase	change $\sqcup$ to 0	change $\sqcup$ to 1

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	$\boxed{1}$	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	$\boxed{1}$	0	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{0}$	1	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{1}$	$\sqcup$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	1	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	0	$\boxed{1}$	$\sqcup$	$\sqcup$	$\cdots$
q_1	$\triangleright$	1	1	0	$\sqcup$	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$

mvRt.tm	s	q	q_0	q_1
0	$s, 0, \rightarrow$	$q_0, \sqcup, \rightarrow$		
1	$s, 1, \rightarrow$	$q_1, \sqcup, \rightarrow$		
$\sqcup$	$q, \sqcup, \leftarrow$		$s, 0, \leftarrow$	$s, 1, \leftarrow$
$\triangleright$	$s, \triangleright, \rightarrow$	$h, \triangleright, -$		

s	$\boxed{\triangleright}$	1	1	0	1	$\sqcup$	$\sqcup$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		
q_1	$\triangleright$	1	1	0	$\sqcup$	$\boxed{\sqcup}$	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	0	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
q	$\triangleright$	1	1	$\boxed{0}$	$\sqcup$	1	$\sqcup$	$\cdots$
q_0	$\triangleright$	1	1	$\sqcup$	$\boxed{\sqcup}$	1	$\sqcup$	$\cdots$
s	$\triangleright$	1	1	$\boxed{\sqcup}$	0	1	$\sqcup$	$\cdots$
$\vdots$		$\vdots$		$\vdots$		$\vdots$		
q	$\boxed{\triangleright}$	$\sqcup$	1	1	0	1	$\sqcup$	$\cdots$
h	$\boxed{\triangleright}$	$\sqcup$	1	1	0	1	$\sqcup$	$\cdots$

TM History

Ancient Greece: Axiomatization of Geometry

Early 19th Century: Non-Euclidean Geometry, Independence of Parallel Postulate (Gauss, Bolyai, Lobachevsky)

Later 19th Century: Rigorous Foundation of Calculus, Real Analysis

1901: Hilbert proposes complete axiomatization of all mathematics, which would reduce all proof to mechanical procedure

1930's: Active interest in the question of what exactly a "mechanical procedure" might be

Formal Models for Mechanical Procedures:

Church: Lambda calculus

Gödel: Recursive function

Kleene: Formal system

Markov: Markov algorithm

Post: Post machine

Turing: Turing machine

Theorem: If A and B are any two of the systems above, and f is a function (say, from bit strings to bits), then f is formalizable in A iff f is formalizable in B.

Church-Turing Thesis: The intuitive idea of "effectively computable" is captured by the precise mathematical definition of "computable" in any of the above models.

"Why is a Turing machine as powerful as any other computational model?"

Intuitive answer: Imagine any computational device. It has:

- Finitely many states
- Ability to scan limited amount per step: one page at a time
- Ability to print limited amount per step: one page at a time
- Next state *determined* by current state and page currently being read (but what about randomization?)

Note: Without the potentially infinite supply of tape cells, paper, extra disks, extra tapes, etc. we have just a (potentially huge) **finite state machine**.

The PC on your desk, with 20 GB of hard disk is a finite state machine with over $2^{160,000,000,000}$ states!

This is better modeled as a TM with a bounded number of states, and an "infinite tape", actually meaning a finite memory that expands whenever necessary.

We have so far defined the *behavior* of a Turing machine – what it will do on a particular input. Now we must define its **semantics** – the way we assign meaning to its behavior.

A Turing machine, once started, may or may not eventually halt. It could fail to halt in a number of ways: run off the left end of the tape, enter a loop of repeated identical configurations, or keep expanding the area of tape it uses forever. If it *does* halt, we want to define what its completed computation means.

One semantics dating back to Turing's original work is to say that the Turing machine **accepts** its input if it halts, and **rejects** its input if it doesn't halt. The language of the machine is then defined to be the set of strings that it accepts.

While simple and useful for some purposes, this semantics doesn't allow us to distinguish among always-halting computations, which after all are our main area of interest.

We will *design* our Turing machines to have understandable behavior. In particular, we will design them to compute **functions** from strings to strings in a particular format:

$$M(w) \;\equiv\; \begin{cases} y & \text{if } M \text{ on input ``}\triangleright w \sqcup\text{'' eventually} \\ & \text{halts with output ``}\triangleright y \sqcup\text{''} \\ \nearrow & \text{otherwise} \end{cases}$$

$\Sigma_0 \;\equiv\; \Sigma - \{\triangleright, \sqcup\}$

Usually, $\Sigma_0 = \{0, 1\}$

Definition 7.1 Let $f : \Sigma_0^\star \to \Sigma_0^\star$ be a total or partial function. We say that f is **recursive** iff $\exists$ TM M, $f = M(\cdot)$, i.e.,

$$(\forall w \in \Sigma_0^\star) \quad f(w) = M(w) \; . \qquad \spadesuit$$

Remark 7.2 *There is an easy to compute, 1:1 and onto map between* $\{0,1\}^\star$ *and* **N**. *Thus we can think of the contents of a TM tape as a natural number and talk about* $f : \mathbf{N} \rightarrow \mathbf{N}$ *being* **recursive**. *(We may visit this issue in HW#3.)*

A partial function $f : \mathbf{N} \rightarrow \mathbf{N}$ is a total function $f : D \rightarrow \mathbf{N}$ where $D \subseteq \mathbf{N}$. A partial function that is not total is called **strictly partial**. If $n \in \mathbf{N} - D$, $f(n) = \nearrow$.

Definition 7.3 Let $S \subseteq \Sigma_0^\star$ or $S \subseteq \mathbf{N}$.

S is a *recursive set* iff the function χ_S is a (total) recursive function,

$$\chi_S(x) \;=\; \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

S is a *recursively enumerable set* (S is r.e.) iff the function p_S is a (partial) recursive function,

$$p_S(x) \;=\; \begin{cases} 1 & \text{if } x \in S \\ \nearrow & \text{otherwise} \end{cases}$$

♠

There is also a common alternate terminology for these two concepts:

- Recursive sets are called **Turing decidable** because an always-halting TM can be designed to output 1 for inputs in the set and 0 for inputs not in it
- Recursively enumerable sets are called **Turing acceptable** because of the semantics mentioned above – a TM can be designed to halt on inputs in the set and not halt on inputs not in it
- The word **enumerable** is from another semantics – a set is r.e. iff a TM can be designed that will list all the elements of the set, running forever if the set is infinite

Proposition 7.4 *If S is recursive then S is r.e.*

Proof: Suppose S is recursive and let M be the TM computing χ_S.

Build M' simulating M but diverging if $M(x) = 0$. Thus M' computes p_S. ♠

We will see that the converse of this proposition is *false*, as there are sets that are r.e. without being recursive.

Proposition 7.5 *The following functions are recursive. They are all total except for p_{even}.*

$$
\begin{aligned}
\mathit{copy}(w) &= ww \\
\sigma(n) &= n+1 \\
\mathit{plus}(n,m) &= n+m \\
\mathit{times}(n,m) &= n \times m \\
\mathit{exp}(n,m) &= n^m \\
\chi_{even}(n) &= \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases} \\
p_{even}(n) &= \begin{cases} 1 & \text{if } n \text{ is even} \\ \nearrow & \text{otherwise} \end{cases}
\end{aligned}
$$

Proof: Exercise: please convince yourself that you can build TMs to compute all of these functions! ♠

If $\mathcal{C}$ is any class of sets, define co-$\mathcal{C}$ to be the class of sets whose complements are in $\mathcal{C}$,

$$\text{co-}\mathcal{C} \quad = \quad \{S \mid \overline{S} \in \mathcal{C}\}$$

Theorem 7.6 *S is recursive iff S and $\overline{S}$ are both r.e.*

Thus, **Recursive** $=$ **r.e.** $\cap$ *co*-**r.e.**

Proof:

($\subseteq$ direction)

If $S \in$ **Recursive** then χ_S is a recursive function by the definition.

Therefore $\chi_{\overline{S}}(x) = 1 - \chi_S(x)$ is also a recursive function.

Thus, S and $\overline{S}$ are both recursive and thus both are r.e.

(other direction)

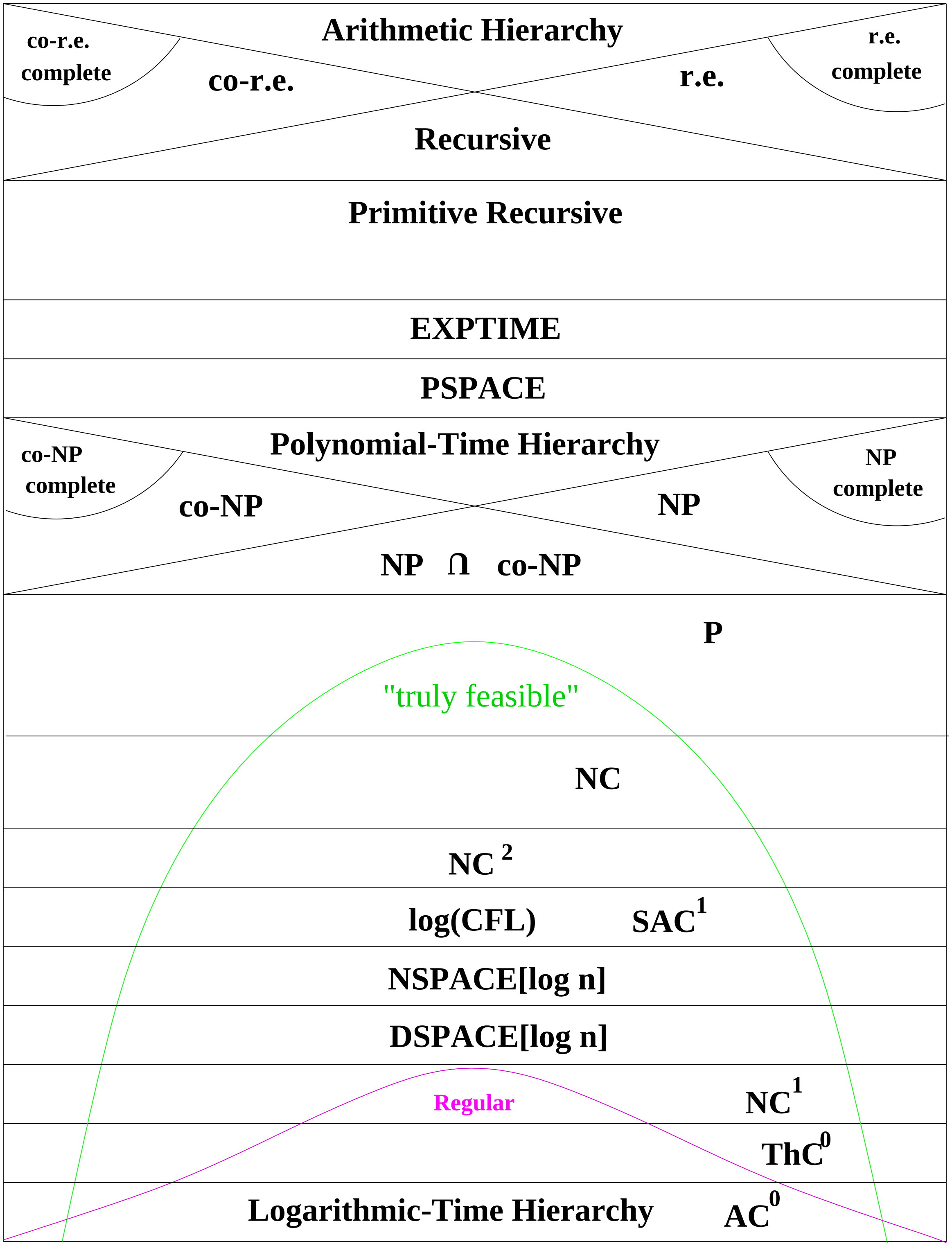
Suppose $S \in$ **r.e.** $\cap$ co-**r.e.**.

By the definition two machines M and M' exist, such that for all inputs x, $p_S(x) = M(x)$ and $p_{\overline{S}}(x) = M'(x)$

We define a new machine T that runs M and M' in parallel. On input x, T does:

1. **for** $n := 1$ to ∞ $\{$
2. run $M(x)$ for n steps.
3. **if** $M(x) = 1$ in n steps **then return**(1)
4. run $M'(x)$ for n steps.
5. **if** $M'(x) = 1$ in n steps **then return**(0)$\}$

Thus, $T(x) = \chi_S(x)$, χ_S is a recursive function, and thus $S \in$ **Recursive**. ♠


Arithmetic Hierarchy
co-r.e. complete
co-r.e.
r.e.
r.e. complete
Recursive
Primitive Recursive
EXPTIME
PSPACE
Polynomial-Time Hierarchy
co-NP complete
co-NP
NP
NP complete
NP ∩ co-NP
P
"truly feasible"
NC
NC 2
log(CFL)
SAC1
NSPACE[log n]
DSPACE[log n]
Regular
NC1
ThC0
Logarithmic-Time Hierarchy
AC0