Savitch's Theorem: For $s(n) \geq \log n$,

$$\textbf{NSPACE}[s(n)] \quad \subseteq \quad \textbf{DSPACE}[(s(n))^2]$$

Immerman-Szelepcsényi Theorem: For $s(n) \geq \log n$,

$$\textbf{NSPACE}[s(n)] \quad = \quad \text{co-}\textbf{NSPACE}[s(n)]$$

Closure Theorem: Virtually all the classes we've considered are closed downward under logspace reductions.

Exercise (HW#6): Logspace reductions are transitive, i.e., if $A \leq B$ and $B \leq C$ then $A \leq C$.

Consider the input (the object we are working on) to be a finite logical structure, e.g., a binary string, a graph, a relational database, or whatever. Remember that a structure includes a list of the objects and lookup tables for all the variables, constants, relations and functions.

Definition 18.1 FO is the set of first-order definable decision problems on finite structures. Let $S \subseteq \mathrm{STRUC}_{\mathrm{fin}}[\Sigma]$.

$S \in \mathrm{FO}$ iff

$$S = \{\mathcal{A} \in \mathrm{STRUC}_{\mathrm{fin}}[\Sigma] \mid \mathcal{A} \models \varphi\}, \quad \text{some } \varphi \in \mathcal{L}(\Sigma)$$

Addition $\qquad Q_+ : \mathrm{STRUC}[\Sigma_{AB}] \rightarrow \mathrm{STRUC}[\Sigma_s]$

$$
\begin{array}{llccccc}
A & & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
B & + & b_1 & b_2 & \ldots & b_{n-1} & b_n \\
\hline
S & & s_1 & s_2 & \ldots & s_{n-1} & s_n
\end{array}
$$

$$
\begin{aligned}
C(i) \equiv\ & (\exists j > i)(A(j) \wedge B(j) \ \wedge \\
& \qquad (\forall k . j > k > i)(A(k) \vee B(k)))
\end{aligned}
$$

$$Q_+(i) \equiv A(i) \oplus B(i) \oplus C(i)$$

$$Q_+(c) \in \mathrm{FO}$$

Encode structures $\mathcal{A} \in \text{STRUC}_{\text{fin}}[\Sigma]$ as binary strings, $\text{bin}(\mathcal{A})$.

Example:

- binary strings: $\text{bin}(\mathcal{A}_w) = w$
- graphs: $G = (\{1, \ldots, n\}, E, s, t)$
  $$\text{bin}(G) = a_{11}a_{12}\ldots a_{nn}s_1s_2\ldots s_{\log n}t_1\ldots t_{\log n}$$

Theorem 18.2 FO $\subseteq$ **L** $=$ **DSPACE**$[\log n]$

Proof:

Given: $\varphi \quad \equiv \quad (\exists x_1)(\forall x_2) \cdots (\forall x_{2k})\psi$

Build **DSPACE**$[\log n]$ TM M s.t.,

$$\mathcal{A} \models \varphi \qquad \Leftrightarrow \qquad M(\text{bin}(\mathcal{A})) = 1$$

By induction on k.

Base case: $k = 0$.

$\varphi \equiv E(s,t)$

$\varphi \equiv s \leq t$

Inductive step:

$$\varphi' \quad \equiv \quad (\exists x_3)(\forall x_4) \cdots (\forall x_{2k})\psi$$

By inductive assumption, there is logspace TM M',

$$\mathcal{A} \models \varphi' \qquad \Leftrightarrow \qquad M'(\text{bin}(\mathcal{A})) = 1$$

Modify M' by adding $2\lceil \log n \rceil$ worktape cells.

Worktape of M:

x_1	x_2	Worktape of M'
$\underbrace{\qquad}_{\lceil \log n \rceil}$	$\underbrace{\qquad}_{\lceil \log n \rceil}$	

M cycles through all values of x_1 until it finds one such that for all x_2, M' accepts. ♠

A Java program can easily be written to test whether $\mathcal{A} \models \varphi$. It has nested `for` loops, one for each quantifier. Since it uses only a constant number of variables of $\log n$ bits each, it represents a deterministic logspace algorithm.

Second-order logic consists of first-order logic, plus new relation variables over which we may quantify.

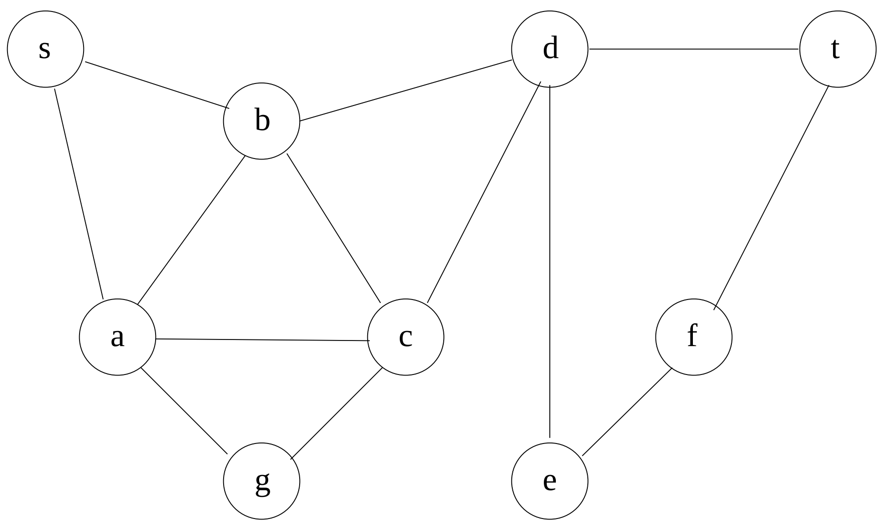
$$(\forall A^r)\varphi$$

For all choices of the r-ary relation A, φ holds.

SO is the set of second-order expressible boolean queries.

SO$\exists$ is the set of second-order existential boolean queries.

$$\Phi_{\text{3-color}} \equiv (\exists R^1)(\exists Y^1)(\exists B^1)(\forall x)[(R(x) \vee Y(x) \vee B(x)) \wedge (\forall y)(E(x,y) \rightarrow \neg(R(x) \wedge R(y)) \wedge \neg(Y(x) \wedge Y(y)) \wedge \neg(B(x) \wedge B(y)))]$$

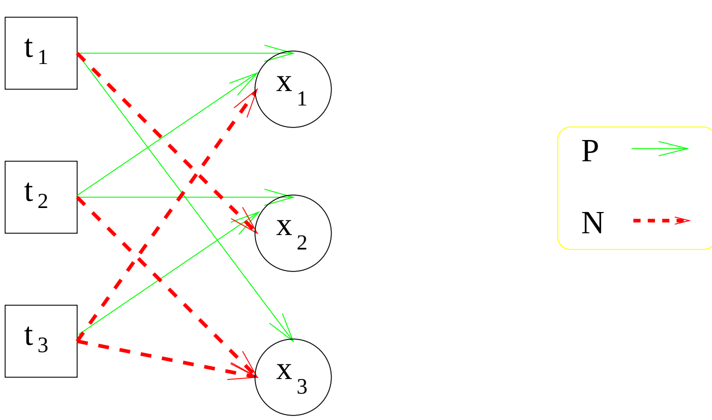


SAT is the set of boolean formulas in conjunctive normal form (CNF) that admit a satisfying assignment.

$$\Phi_{\text{SAT}} \equiv (\exists S^1)(\forall t)(\exists x)(C(t) \rightarrow (P(t,x) \wedge S(x)) \vee (N(t,x) \wedge \neg S(x)))$$

$$\begin{aligned} C(t) &\equiv \text{“}t \text{ is a clause; otherwise } t \text{ is a variable.”} \\ P(t,x) &\equiv \text{“Variable } x \text{ occurs positively in clause } t\text{.”} \\ N(t,x) &\equiv \text{“Variable } x \text{ occurs negatively in clause } t\text{.”} \end{aligned}$$

$$\varphi \equiv (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$$



CLIQUE is the set of pairs $\langle G, k\rangle$ such that G is a graph that has a complete subgraph of size k.

Let $\mathrm{Inj}(f)$ mean that f is an injective function

$$\mathrm{Inj}(f) \;\equiv\; (\forall xy)(f(x) = f(y) \;\rightarrow\; x = y)$$

$$\Phi_{\text{CLIQUE}} \;\equiv\; (\exists f^1.\mathrm{Inj}(f))(\forall xy)((x \neq y \wedge f(x) < k \wedge f(y) < k) \\ \rightarrow \quad E(x, y))$$

Theorem 18.3 (Fagin's Theorem) NP *is equal to the set of existential, second-order boolean queries,* **NP** $=$ SO$\exists$.

Proof: NP $\supseteq$ SO$\exists$: We are given a second-order existential sentence

$$\Phi \equiv (\exists R_1^{r_1}) \ldots (\exists R_k^{r_k})\psi \ \in \ \mathcal{L}(\Sigma)$$

Build NP machine N s.t. for all $\mathcal{A} \in \text{STRUC}_{\text{fin}}[\Sigma]$,

$$\mathcal{A} \models \Phi \quad \Leftrightarrow \quad N(\text{bin}(\mathcal{A})) = 1 \qquad (18.4)$$

$\mathcal{A} \in \text{STRUC}_{\text{fin}}[\Sigma], \qquad n = \|\mathcal{A}\|.$

N nondeterministically writes down a binary string of length n^{r_1} representing R_1, and similarly for R_2 through R_k.

$$\mathcal{A}' = (\mathcal{A}, R_1, R_2, \ldots, R_k)$$

N accepts iff $\mathcal{A}' \models \psi$.

Since FO $\subseteq$ **L** (Th 19.2) we can test if $\mathcal{A}' \models \psi$ in logspace and so certainly in NP. Thus Equivalence 18.4 holds.

NP $\subseteq$ SO$\exists$: Let N be an **NTIME**$[n^k]$ TM.

Write an SO$\exists$ sentence,

$$\Phi \quad = \quad (\exists C_0^{2k} \ldots C_{g-1}^{2k} \Delta^k)\varphi \tag{18.5}$$

meaning, "There exists an accepting computation $\overline{C}, \Delta$ of N."

We will show that:

$$\mathcal{A} \models \Phi \quad \Leftrightarrow \quad N(\text{bin}(\mathcal{A})) = 1$$

Remark 18.6 *Assume that language has numeric relations: $\leq, \text{SUC}$ and constants $0, max$ refering to total ordering on the universe, its successor relation, the minimum and maximum elements in this ordering, respectively.*

Then φ in Equation 18.5 can be made universal,

$$\varphi \quad \equiv \quad (\forall x_1 \cdots x_t)\psi,$$

with ψ quantifier free.

$$\text{Fix } \mathcal{A}, \quad n = \|\mathcal{A}\|$$

Possible contents of a computation cell for N:

$$\Gamma = \{\gamma_0, \ldots, \gamma_{g-1}\} = (Q \times \Sigma) \cup \Sigma$$

$C_i(s_1, \ldots, s_k, t_1, \ldots, t_k)$ means cell $\bar{s}$ at time $\bar{t}$ is symbol γ_i

$\Delta(\bar{t})$ means the $\bar{t} + 1^{\text{st}}$ step of the computation makes choice "1"; otherwise it makes choice "0".

	Space							
	0	1	$\bar{s}$	$n-1$	n		n^k-1	Δ
Time 0	$\langle q_0, w_0 \rangle$	w_1	$\cdots$	w_{n-1}	$\sqcup$	$\cdots$	$\sqcup$	δ_0
1	w_0	$\langle q_1, w_1 \rangle$	$\cdots$	w_{n-1}	$\sqcup$	$\cdots$	$\sqcup$	δ_1
	$\vdots$	$\vdots$	$\vdots$			$\vdots$		$\vdots$
$\bar{t}$			a_{-1} a_0 a_1					δ_t
$\bar{t}+1$			b					δ_{t+1}
	$\vdots$	$\vdots$	$\vdots$			$\vdots$		$\vdots$
n^k-1	$\langle q_f, 1 \rangle$	$\sqcup$	$\cdots$	$\sqcup$	$\sqcup$	$\cdots$	$\sqcup$	

Accepting computation of N on input $w_0 w_1 \cdots w_{n-1}$

Write first-order sentence, $\varphi(\overline{C}, \Delta)$, saying that $\overline{C}, \Delta$ codes a valid accepting computation of N.

$$\varphi \quad \equiv \quad \alpha \wedge \beta \wedge \eta \wedge \zeta$$

$\alpha \equiv$ row 0 codes input bin$(\mathcal{A})$

$\beta \equiv (\forall \bar{s}, \bar{t}, i \neq j)(\neg(C_i(\bar{s}, \bar{t}) \wedge C_j(\bar{s}, \bar{t})))$

$\eta \equiv (\forall \bar{t})$(row $\bar{t} + 1$ follows from row $\bar{t}$ via move $\Delta(\bar{t})$ of N)

$\zeta \equiv$ last row of computation is accept ID

$$\mathcal{A} \models \Phi \quad \Leftrightarrow \quad N(\text{bin}(\mathcal{A})) = 1$$

$$\Phi \equiv \exists C_0^{2k} C_1^{2k} \cdots C_{g-1}^{2k} \Delta^k(\varphi)$$

$\equiv$ "$\exists$ an accepting compution: N(me) = 1"

$$\alpha \quad \equiv \quad \text{row 0 codes input bin}(\mathcal{A})$$

Assume Σ has only single unary relation symbol, R.

0	1		$n-1$	n		$n^k - 1$
$\langle q_0, w_0 \rangle$	w_1	$\cdots$	w_{n-1}	$\sqcup$	$\cdots$	$\sqcup$

$$\gamma_0 = 0;\ \gamma_1 = 1;\ \gamma_2 = \sqcup;\ \gamma_3 = \langle q_0, 0 \rangle;\ \gamma_4 = \langle q_0, 1 \rangle$$

$$\begin{aligned}
\alpha \ \equiv & \quad R(0) \to C_4(\bar{0}, \bar{0}) \\
\wedge & \quad \neg R(0) \to C_3(\bar{0}, \bar{0}) \\
\wedge & \quad (\forall i > 0)(R(i) \to C_1(\bar{0}i, \bar{0}) \\
& \qquad\qquad \wedge \neg R(i) \to C_0(\bar{0}i, \bar{0})) \\
\wedge & \quad (\forall \bar{s} \geq n) C_2(\bar{s}, \bar{0})
\end{aligned}$$

Most interesting case: η

$$\langle a_{-1}, a_0, a_1, \delta \rangle \stackrel{N}{\to} b$$

Triple a_{-1}, a_0, a_1 leads to b via move δ of N.

$$\eta_1 \quad \equiv$$

$$(\forall \bar{t}. \bar{t} < \overline{max})(\forall \bar{s}. \bar{0} < \bar{s} < \overline{max})$$

$$\bigwedge_{\langle a_{-1}, a_0, a_1, \delta \rangle \stackrel{N}{\to} b} (\neg^{\delta} \Delta(\bar{t}) \vee$$

$$\neg C_{a_{-1}}(\bar{s}-1, \bar{t}) \vee \neg C_{a_0}(\bar{s}, \bar{t}) \vee \neg C_{a_1}(\bar{s}+1, \bar{t}) \vee C_b(\bar{s}, \bar{t}+1))$$

Here $\neg^{\delta}$ is $\neg$ if $\delta = 1$ and it is the empty symbol if $\delta = 0$.

$$\eta \quad \equiv \quad \eta_0 \ \wedge \ \eta_1 \ \wedge \ \eta_2$$

where η_0 and η_2 encode the same information when $\bar{s} = \bar{0}$ and $\overline{max}$ respectively. ♠

Theorem 18.7 (Cook-Levin Theorem)

SAT *is* **NP**-*complete.*

(This theorem was proved roughly simultaneously by Steve Cook in the USA and Leonid Levin in the USSR, before Fagin proved his theorem. We'll prove Cook-Levin as a corollary of Fagin's Theorem, somewhat contrary to history. But note that the proof of Cook-Levin in Sipser, for example, is almost the same as our proof of Fagin.)

Proof: Let $B \in \mathbf{NP}$. By Fagin's theorem,

$$B = \{\mathcal{A} \mid \mathcal{A} \models \Phi\}$$

$$\Phi = (\exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k)(\forall x_1 \cdots x_t)\psi(\bar{x})$$

with ψ quantifier-free and CNF,

$$\psi(\bar{x}) \quad = \quad \bigwedge_{j=1}^{r} T_j(\bar{x})$$

with each T_j a disjunction of literals.

Let $\mathcal{A}$ be arbitrary, $\quad n = \|\mathcal{A}\|$

Define formula $\varphi(\mathcal{A})$ as follows:

boolean variables:

$C_i(e_1, \ldots, e_{2k}), \Delta(e_1, \ldots, e_k), \qquad i = 1, \ldots, g, e_1, \ldots, e_{2k} \in |\mathcal{A}|$

clauses:

$$T_j(\bar{e}), \quad j = 1, \ldots, r, \bar{e} \in |\mathcal{A}|^t$$

$T'_j(\bar{e})$ is $T_j(\bar{e})$ with atomic numeric or input predicates, $R(\bar{e})$, replaced by **true** or **false** according as they are true or false in $\mathcal{A}$. Occurrences of $C_i(\bar{e})$, and $\Delta(\bar{e})$ are considered boolean variables.

$$\Phi \;\equiv\; (\exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k)(\forall x_1 \cdots x_t) \bigwedge_{j=1}^{r} T_j(\bar{x})$$

$$\varphi(\mathcal{A}) \;\equiv\; \bigwedge_{e_1,\ldots,e_t \in |\mathcal{A}|} \bigwedge_{j=1}^{r} T'_j(\bar{e})$$

$\mathcal{A} \in B \quad \Leftrightarrow \quad \mathcal{A} \models \Phi \quad \Leftrightarrow \quad \varphi(\mathcal{A}) \in \mathrm{SAT}$♠

Proposition 18.8

3-SAT $=$ $\{\varphi \in$ CNF-SAT $\mid \varphi$ *has* ≤ 3 *literals per clause*$\}$

3-SAT *is* **NP**-*complete.*

Proof: Show SAT $\leq$ 3-SAT.

Example:

$$C \quad = \quad (\ell_1 \vee \ell_2 \vee \cdots \vee \ell_7)$$

$$C' \equiv (\ell_1 \vee \ell_2 \vee d_1) \wedge (\overline{d_1} \vee \ell_3 \vee d_2) \wedge (\overline{d_2} \vee \ell_4 \vee d_3) \wedge (\overline{d_3} \vee \ell_5 \vee d_4) \wedge (\overline{d_4} \vee \ell_6 \vee \ell_7)$$

Claim: $C \in$ SAT $\Leftrightarrow$ $C' \in$ 3-SAT

In general, just do this construction for each clause independently, introducing separate dummy variables for each cluase. The AND of all the new 3-variable clauses is satisfiable iff the AND of all the old clauses is. ♠