First-Order Proof Rule:

Modus Ponens (M.P.)
$\rightarrow$ Elim

$$\frac{\Gamma \vdash \varphi \rightarrow \psi, \ \Gamma \vdash \varphi}{\Gamma \vdash \psi}$$

First-Order Axioms:

all generalizations of the following:

0	Tautologies on **at most three** boolean variables
1a	$t = t$
1b	$(t_1 = t_1' \wedge \cdots \wedge t_k = t_k') \rightarrow f(t_1, \ldots, t_k) = f(t_1', \ldots, t_k')$
1c	$(t_1 = t_1' \wedge \cdots \wedge t_k = t_k') \rightarrow (R(t_1, \ldots, t_k) \rightarrow R(t_1', \ldots, t_k'))$
2	$(\forall x)(\varphi) \rightarrow \varphi[x \leftarrow t]$
3	$\varphi \rightarrow (\forall x)(\varphi)$, x not free in φ
4	$(\forall x)(\varphi \rightarrow \psi) \ \rightarrow \ ((\forall x)(\varphi) \rightarrow (\forall x)(\psi))$

Prop. 12.7: M.P. preserves truth, validity, and semantic implication.

Props. 12.8, 12.9, 12.10, 12.12, 12.14, 12.15: Every instance of every axiom from [P] is valid.

Soundness Theorem: Every first-order theorem is valid, i.e., for all Σ, and for all $\varphi \in \mathcal{L}(\Sigma)$,

$$\vdash \varphi \models \varphi$$

Furthermore, first-order proofs preserve truth, and semantic implication, i.e., for all $\Gamma \subseteq \mathcal{L}(\Sigma)$,

$$\textbf{if} \;\; \Gamma \vdash \varphi \quad \textbf{then} \;\; \Gamma \models \varphi$$

Proof: The axioms are valid and M.P. preserves validity.
♠

Prop. 13.9:

$$\text{FO-THEOREMS} \; = \; \{\varphi \mid \vdash \varphi\} \; \in \; \textbf{r.e.}$$

Two Proof Systems

The proof rules of Fitch preserve not only *validity* but *provability* in [P]'s system. That is, the existence of a Fitch proof *proves* the existence of a [P] proof.

We establish this fact by proving *metatheorems*:

- **Generalization:** If $\Gamma \vdash \varphi$ and x does not occur freely in Γ, then $\Gamma \vdash \forall x(\varphi)$
- **Deduction:** If $\Gamma \cup \{\varphi\} \vdash \psi$, then $\Gamma \vdash \varphi \to \psi$
- **Proof By Contradiction:** If $\Gamma \cup \{\varphi\} \vdash \bot$, then $\Gamma \vdash \neg\varphi$
- **Add A Constant:** If $\Gamma \vdash \exists x(\varphi)$ and $\Gamma, \varphi[x \leftarrow c] \vdash \psi$, where c does not occur in Γ, φ, or ψ, then $\Gamma \vdash \psi$

Proposition 14.1 $\vdash \forall xyz((x = y \wedge y = z) \rightarrow x = z)$

Proof:

1.	$\{(x = y \wedge y = z)\} \vdash x = y$	$\wedge$ Elim
2.	$\{(x = y \wedge y = z)\} \vdash y = z$	$\wedge$ Elim
3.	$\vdash x = y \rightarrow y = x$	Lect. 13, sl. 12
4.	$\{(x = y \wedge y = z)\} \vdash y = x$	MP 1, 3
5.	$\vdash z = z$	AX 1a
6.	$\{(x = y \wedge y = z)\} \vdash (y = x \wedge z = z)$	$\wedge$ Intro 4, 5
7.	$\vdash (y = x \wedge z = z) \rightarrow (y = z \rightarrow x = z)$	AX 1c
8.	$\{(x = y \wedge y = z)\} \vdash (y = z \rightarrow x = z)$	MP 6, 7
9.	$\{(x = y \wedge y = z)\} \vdash x = z$	MP 2, 8
10.	$\vdash (x = y \wedge y = z) \rightarrow x = z$	$\rightarrow$ Intro 9
11.	$\vdash \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 10
12.	$\vdash \forall y \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 11
13.	$\vdash \forall x \forall y \forall z((x = y \wedge y = z) \rightarrow x = z)$	$\forall$ Intro. 12

Definition 14.2 Γ is *consistent* iff $\Gamma \nvdash \bot$ ♠

[Already shown:

Soundness Th: If Γ is satisfiable Then Γ is consistent.]

Completeness Theorem:

If Γ is consistent then Γ is satisfiable.

Proof: Let $\Gamma \subseteq \mathcal{L}(\Sigma)$ be consistent.

We will build $\mathcal{A}$ such that $\mathcal{A} \models \Gamma$.

Idea: Extend Γ to a maximally consistent set of formulas Δ.

Δ answers every question and thus defines a model $\mathcal{A}$.

Let $c_0, c_1, c_2, \ldots$ be new constant symbols.

$$\Sigma' \;=\; \Sigma \;\cup\; \{c_i \;|\; i \in \mathbf{N}\}$$

$$\mathcal{L}(\Sigma') \;=\; \{\varphi_0, \varphi_1, \varphi_2, \ldots\}$$

c_n does not occur in $\varphi_0, \varphi_1, \ldots, \varphi_n$

$$\Gamma_{-1} \;=\; \Gamma$$

$$\Gamma_n \;=\; \Gamma_{n-1} \;\cup\; \begin{cases} \varphi_n & \text{if } \Gamma_{n-1} \cup \{\varphi_n\} \text{ consistent} \\ \neg\varphi_n & \text{otherwise} \end{cases}$$

$$\cup \quad \{\psi(c_n) \;|\; (\exists v)\psi(v) \text{ just added}\}$$

Each new statement is either already forced to be true, forced to be false, or not yet forced. In the last case we make it true. If the new true statement says that an element exists with a property ψ, we make c_n that element.

Claim: Each Γ_n is consistent.

Proof: By induction on n.

Base case: Γ_{-1} consistent by assumption.

Inductive step: Assume Γ_{n-1} consistent.

At least one of $\Gamma_{n-1} \cup \{\varphi_n\}$, $\Gamma_{n-1} \cup \{\neg\varphi_n\}$ is consistent.

If $\Gamma_{n-1} \cup \{(\exists v)(\psi(v))\}$ consistent

Then so is $\Gamma_{n-1} \cup \{(\exists v)(\psi(v)), \psi(c_n)\}$ by $\exists$ Elimination.

♠

$$\Gamma = \Gamma_{-1} \subseteq \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$

$$\Delta \equiv \bigcup_{n=0}^{\infty} \Gamma_n$$

Δ is consistent and complete and has the *Henkin property* (that every provable $\exists$ statement has a constant witness):

Consistent: Suppose $\Delta \vdash \perp$

Since proofs are finite, some $\Gamma_i \vdash \perp$

Complete: for all $\varphi_n \in \mathcal{L}(\Sigma')$, $\Delta \vdash \varphi_n$ or $\Delta \vdash \neg\varphi_n$.

Henkin: if $\Delta \vdash (\exists v)\psi(v)$ **then** for some constant k, $\Delta \vdash \psi(k)$.

Definition 14.3 $c_i \equiv c_j$ iff $\Delta \vdash c_i = c_j$ ♠

Claim: $\equiv$ is an equivalence relation.

Proof: $\vdash c_i = c_i$, $\vdash c_i = c_j \to c_j = c_i$,

$\vdash (c_i = c_j \wedge c_j = c_k) \to c_i = c_k$ ♠

$$|\mathcal{A}| = \{[c_i] \mid i \in \mathbf{N}\}$$

$$x^{\mathcal{A}} = [c_n], \quad \text{s.t. } \Delta \vdash x = c_n$$

$$R^{\mathcal{A}} = \{\langle [c_{i_1}], \ldots, [c_{i_{r(R)}}]\rangle \mid \Delta \vdash R(c_{i_1}, \ldots, c_{i_{r(R)}})\}$$

$$f^{\mathcal{A}} = \{\langle [c_{i_1}], \ldots, [c_{i_{r(f)+1}}]\rangle \mid \Delta \vdash f(c_{i_1}, \ldots, c_{i_{r(f)}}) = c_{i_{r(f)+1}}\}$$

Note: $f^{\mathcal{A}}$ is well defined by AX1b, and $R^{\mathcal{A}}$ is well defined by AX1c.

Claim 14.4 $\mathcal{A} \models \Delta$

Proof:

First show, $t_1^{\mathcal{A}} = t_2^{\mathcal{A}} \quad \Leftrightarrow \quad \Delta \vdash t_1 = t_2$
by induction on $|t_1| + |t_2|$.

base case: $t_1 = u; t_2 = v$. $t_1^{\mathcal{A}} = [c_1], t_2^{\mathcal{A}} = [c_2]$,
where $\Delta \vdash t_1 = c_1, t_2 = c_2$.

$$\begin{aligned}(t_1^{\mathcal{A}} = t_2^{\mathcal{A}}) &\Leftrightarrow ([c_1] = [c_2]) \\ &\Leftrightarrow \Delta \vdash c_1 = c_2 \\ &\Leftrightarrow \Delta \vdash t_1 = t_2\end{aligned}$$

inductive case: $t_1 = f(s_1, \ldots, s_a)$;

$$s_i^{\mathcal{A}} = [k_i],\ i = 1, \ldots, a; \quad t_2^{\mathcal{A}} = [c_2]$$

By inductive hypothesis: $\Delta \vdash t_2 = c_2$;

$$\Delta \vdash s_i = k_i, \quad i = 1, \ldots, a$$

$$\begin{aligned}
(t_1^{\mathcal{A}} = t_2^{\mathcal{A}}) \ &\Leftrightarrow\ (f^{\mathcal{A}}([k_1], \ldots, [k_a]) = [c_2]) \\
&\Leftrightarrow\ \Delta \vdash f(k_1, \ldots, k_a) = c_2 \\
&\Leftrightarrow\ \Delta \vdash f(s_1, \ldots, s_a) = c_2 \\
&\Leftrightarrow\ \Delta \vdash t_1 = t_2
\end{aligned}$$

Now, by induction on $\varphi \in \mathcal{L}(\Sigma')$ show

$$\mathcal{A} \models \varphi \quad \Leftrightarrow \quad \Delta \vdash \varphi$$

Base case: $\varphi = R(t_1, \ldots, t_{r(R)})$

Inductive case: $\varphi = \neg\psi$

Inductive case: $\varphi = \alpha \vee \beta$

Inductive case: $\varphi = (\forall v)\psi(v)$

This completes the proof of Claim 14.4. ♠

This completes proof of the Completeness Theorem. ♠

Corollary 14.5

$$\Gamma \models \varphi \quad \Leftrightarrow \quad \Gamma \vdash \varphi$$

Proof: Suppose $\Gamma \not\vdash \varphi$.

$$\Gamma, \neg\varphi \not\vdash \bot$$

$$\text{there exists } \mathcal{A}, \quad \mathcal{A} \models \Gamma \cup \{\neg\varphi\}$$

$$\Gamma \not\models \varphi$$

♠

FO-THEOREM $=$ FO-VALID $\in$ **r.e.**

Notation: $\Gamma \vdash \varphi$; $\mathcal{A} \models \varphi$

Theorem 14.6 (Compactness Theorem)

Suppose every finite subset of Γ *has a model.*

Then Γ *has a model*

Proof: If Γ is inconsistent, then some finite subset of Γ is inconsistent because proofs are finite.

No finite subset of Γ is inconsistent.

Γ is consistent

Γ has a model

♠

$$\text{Theory}(\mathbf{N}) \;=\; \{\varphi \in \mathcal{L}(\Sigma_N) \;|\; \mathbf{N} \models \varphi\}$$

$$\Gamma \;=\; \text{Theory}(\mathbf{N}) \;\cup\; \{c > 0, c > 1, c > 2, c > 3, \ldots\}$$

Corollary 14.7 Γ *has a model.*

There is a countable model of Theory($\mathbf{N}$) *that is not isomorphic to* $\mathbf{N}$.

$\mathcal{L}(\Sigma_N)$ *cannot uniquely characterize* $\mathbf{N}$.

Proof: Every finite subset of Γ is satisfiable by $(\mathbf{N}, i)$ for i sufficiently large.

By Compactness, Γ is satisfiable. ♠

Corollary 14.8 *"Connectedness" is not expressible in the first-order language of graphs,* $\mathcal{L}(\Sigma_g)$

Proof:

Suppose that $\chi \equiv$ "I am connected."

$$\Gamma \quad = \quad \{\chi\} \ \cup \ \{\mathrm{DIST}(s,t) > 1, \mathrm{DIST}(s,t) > 2, \ldots\}$$

$$\mathrm{DIST}(x_0, x_n) > n \quad \equiv$$

$$(\forall x_1 \cdots x_{n-1}) \bigvee_{i=0}^{n-1} (x_i \neq x_{i+1} \wedge \neg E(x_i, x_{i+1}))$$

Every finite subset of Γ is satisfiable.

By Compactness, Γ is satisfiable.

$$\Rightarrow\Leftarrow$$

"Connectedness" is not expressible in the first-order language of graphs. ♠