Th. 12.1: For any vocabulary Σ, $\mathcal{W} \in \text{STRUC}[\Sigma]$, $\varphi \in \mathcal{L}(\Sigma)$,

in the game where A asserts and B denies that $\mathcal{W} \models \varphi$,

$$\mathcal{W} \models \varphi \Leftrightarrow A \text{ has a winning strategy}$$

$$\mathcal{W} \not\models \varphi \Leftrightarrow B \text{ has a winning strategy}$$

Question: If φ is a tautology and A asserts φ, A has a winning strategy in the game. But, must A win no matter what he does?

Answer: No, if A plays sufficiently badly, he can lose. For example, suppose $\varphi \equiv (\text{Tet}(a) \vee \neg\text{Tet}(a))$.

A must choose which disjunct he thinks is true. A wrong choice here will lose the game even though $(\text{Tet}(a) \vee \neg\text{Tet}(a))$ is a tautology.

Notation: For $\Gamma \subseteq \mathcal{L}(\Sigma)$, $\varphi \in \mathcal{L}(\Sigma)$, "$\Gamma \vdash \varphi$" is read, "$\Gamma$ proves φ", and means, "There is a first-order proof of φ assuming Γ."

Modus Ponens (M.P.): $\dfrac{\Gamma \vdash \varphi \rightarrow \psi, \ \Gamma \vdash \varphi}{\Gamma \vdash \psi}$

Prop. 12.7: Modus Ponens preserves truth, validity, and semantic implication, i.e.,

if $\mathcal{A} \models \varphi \rightarrow \psi$ and $\mathcal{A} \models \varphi$ **then** $\mathcal{A} \models \psi$.

if $\Gamma \models \varphi \rightarrow \psi$ and $\Gamma \models \varphi$ **then** $\Gamma \models \psi$.

If φ is a first-order formula, then $\forall x(\varphi)$ is called a **generalization** of φ.

Prop. 12.8: If $\models \varphi$, then $\models \forall x(\varphi)$.

all generalizations of the following

AX0: Tautologies on **at most three boolean variables,** with first-order formula substituted for the variables.

1. $x_1 \rightarrow x_1$
2. $x_1 \rightarrow (x_1 \vee x_2)$
3. $x_1 \vee \neg x_1$
4. $x_1 \rightarrow (\neg x_1 \rightarrow x_2)$

1. $(\forall u)(\exists v)E(u,v) \;\rightarrow\; (\forall u)(\exists v)E(u,v)$
2. $(\forall z)(z < z + z) \rightarrow ((\forall z)(z < z + z) \vee (\forall y)(y < z))$
3. $(\exists z)R(z) \;\vee\; \neg(\exists z)R(z)$
4. $\text{prime}(17) \rightarrow (\neg\text{prime}(17) \rightarrow 0 \neq 0)$

Proposition 13.1 *All members of AX0 are valid.*

Equality Axioms

all generalizations of the following

AX1a $t = t$, for any term t

AX1b $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow f(t_1, \ldots, t_k) = f(t'_1, \ldots, t'_k)$ for terms $t_1, \ldots, t'_k$, $f \in \Phi$, $r(f) = k$

Ax1c $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow (R(t_1, \ldots, t_k) \rightarrow R(t'_1, \ldots, t'_k))$ for terms $t_1, \ldots, t'_k$, $R \in \Pi$, $r(R) = k$

Proposition 13.2 *Every instance of AX1 is valid.*

Proof: Because “=” is interpreted as “identically equal”. ♠

Definition 13.3 Term t is *substitutable* for variable x in φ iff no free occurrence of x in φ is within the scope of a quantifier for a variable z occurring in t.

$\varphi[x \leftarrow t]$ is the result of substituting t for all free occurrences of x in φ.

We never use this expression unless t is substitutable for x in φ. ♠

$$\begin{aligned} \alpha &\equiv (\exists y)(y < x) \\ \alpha[x \leftarrow z+1] &\equiv (\exists y)(y < z+1) \\ \alpha[x \leftarrow f(u)+v] &\equiv (\exists y)(y < f(u)+v) \\ \alpha' &\equiv (\exists y)(y < y) \end{aligned}$$

$z+1$, u, $f(u)$, $f(u)+v$ are substitutable for x in α.

y, $y+1$ are not substitutable for x in α.

$$\begin{aligned} \alpha &\equiv \text{“}x \text{ is not the least element”} \\ \alpha[x \leftarrow z+1] &\equiv \text{“}z+1 \text{ is not the least element”} \end{aligned}$$

Instantiation Axioms

all generalizations of the following

AX2: $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t]$, x a variable, t a term, t substitutable for x in φ.

Proposition 13.4 *Every instance of AX2 is valid.*

Proof: Let $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t] \in$ AX2.

♠

Replacing Bound Variables

Lemma 13.5 *Let $\mathcal{A}, \mathcal{A}'$ be identical except for how they interpret some variables not free in φ. Then,*

$$\mathcal{A} \models \varphi \qquad \Leftrightarrow \qquad \mathcal{A}' \models \varphi$$

Proof: By induction on φ.

Base case: $\varphi \equiv R(t_1, \ldots, t_k)$

Inductive case 1: $\varphi \equiv \neg\psi$

Inductive case 2: $\varphi \equiv (\alpha \vee \beta)$

Inductive case 3: $\varphi \equiv \forall x(\psi)$

♠

Generalization Axioms

all generalizations of the following

AX3: $\varphi \to \forall x(\varphi)$, where x does not occur freely in φ.

Proposition 13.6 *Every instance of AX3 is valid.*

Proof: Let $\varphi \to \forall x(\varphi) \in$ AX3.
Let $\mathcal{A} \in$ STRUC$[\Sigma]$ be arbitrary.
Suppose $\mathcal{A} \models \varphi$

$$(|\mathcal{A}|, \mu) \models \forall x(\varphi) \quad \Leftrightarrow \quad (\text{for all } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \models \varphi$$

By Lemma 13.5, $\mathcal{A} \models \forall x(\varphi)$

♠

One Last Set of Axioms

all generalizations of the following

AX4: $\forall x(\varphi \to \psi) \ \to \ (\forall x(\varphi) \to \forall x(\psi))$

Proposition 13.7 *Every instance of AX4 is valid.*

Proof:

$$\forall x(\varphi \to \psi) \ \to \ (\forall x(\varphi) \to \forall x(\psi)) \quad \in \quad \text{AX4}$$

Suppose $\mathcal{A} \models \forall x(\varphi \to \psi)$.
(finish on whiteboard)

♠

Definition 13.8 Let $\Gamma \subseteq \mathcal{L}(\Sigma)$. A *proof in FO logic* from Γ is a finite sequence of formulas,

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \cdots\ \alpha_n$$

such that for each α_i,

1. α_i is an axiom, or
2. $\alpha_i \in \Gamma$, or
3. $(\exists j, k < i)\ \alpha_i$ follows from α_j, α_k by M.P.

If φ is in a proof from Γ, write $\Gamma \vdash \varphi$.

If φ is in a proof from $\emptyset$, write $\vdash \varphi$; φ is a *theorem*.

FO-Theorems $= \{\varphi \mid \vdash \varphi\}$

Proposition 13.9 **FO-Theorems $\in$ r.e.**

Proof:

1. **for** $i := 1$ to ∞ {
2. **for** each string S of length i {
3. **if** (S is a correct, FO proof)
4. **then** output LastLine(S)
5. } }

♠

Proposition 13.10 $\vdash (x = y \to y = x)$

This formula holds in any structure containing an assignment to x and y. We know it's true *because $x = y$ means that $\mu(x) = \mu(y)$ in the structure. But we want to* prove it *in [P]'s formal system.*

Proof:

1.	$(x = y \wedge x = x) \to (x = x \to y = x)$	AX 1c
2.	$x = x$	AX 1a
3.	$x = x \to$ $((x = y \wedge x = x) \to (x = x \to y = x)) \to$ $(x = y \to y = x)$	AX 0
4.	$((x = y \wedge x = x) \to (x = x \to y = x)) \to$ $(x = y \to y = x)$	MP 2, 3
5.	$x = y \to y = x$	MP 1, 4

3: $\alpha \to ((\beta \wedge \alpha) \to (\alpha \to \gamma)) \to (\beta \to \gamma)$

♠

Theorem 13.11 (Soundness Theorem)
If $\vdash \varphi$ *then* $\models \varphi$.

$$\text{FO-THEOREMS} \subseteq \text{FO-VALID}$$

Proof: The axioms are valid and M.P. preserves validity. ♠

Soundness is also called "consistency". It says "everything provable is true", while completeness says "everything true is provable".

Corollary 13.12 *If* $\Gamma \vdash \varphi$ *Then* $\Gamma \models \varphi$.

Proof: The axioms are valid and M.P. preserves truth, i.e.,

If $\Gamma \models \varphi$ and $\Gamma \models \varphi \to \psi$

Then $\Gamma \models \psi$.

Thus, by induction on the length of a proof from Γ, every line is true in every model of Γ. ♠

Two Proof Systems

The main text [P] uses the system we have just described, with one proof rule (Modus Ponens) and lots of axioms. The LPL book uses the system F or Fitch, which has no axioms but lots of proof rules.

We've now seen all the [P] axioms and proved that they are valid – they only prove true statements.

But we'd rather do our proofs in Fitch, because it more closely follows our informal processes and because we have software to help us with it.

So we need to show that Fitch's rules only prove things that are provable in [P]'s system.

(It's also true that Fitch can prove all of [P]'s axioms, and it has MP as a rule ($rightarrow$ Elim) so they prove the same statements.)

Generalization MetaTheorem $=$ $\forall$ **Intro**

If $\Gamma \vdash \varphi$ and x does not occur freely anywhere in Γ

Then $\Gamma \vdash \forall x(\varphi)$

Proof: By induction on the length of the proof of φ from Γ.

Base case: $n = 1$, $\varphi \in$ AXIOMS:

$\forall x(\varphi) \in$ AXIOMS.

Base case: $n = 1$, $\varphi \in \Gamma$.

$\varphi \to \forall x(\varphi)$ $\in$ AX3 since x not free in φ.

φ given

$\forall x(\varphi)$ M.P.

Inductive case: Assume true for all proofs of length less than n.

$\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \varphi$ is a proof from Γ.

By induction, $\Gamma \vdash \forall x(\alpha_i)$, $1 \leq i \leq n-1$.

φ follows from α_i, α_j by M.P., $i, j < n$

$$\alpha_j = (\alpha_i \rightarrow \varphi)$$

1.	$\forall x(\alpha_i)$	
2.	$\forall x(\alpha_i \rightarrow \varphi)$	
3.	$\forall x((\alpha_i \rightarrow \varphi) \rightarrow ((\forall x)(\alpha_i) \rightarrow (\forall x)\varphi))$	AX4
4.	$\forall x(\alpha_i) \rightarrow \forall x(\varphi)$	M.P. 2,3
5.	$\forall x(\varphi)$	M.P. 1,4

♠

The Deduction MetaTheorem $=$ $\rightarrow$ **Intro**

If $\Gamma \cup \{\varphi\} \vdash \psi$

Then $\Gamma \vdash \varphi \rightarrow \psi$

Proof: By induction on n, the length of the proof of ψ from $\Gamma \cup \{\varphi\}$.

Base case: $n = 1$, $\varphi = \psi$:

$$\psi \rightarrow \psi \in \text{AX0}$$

Base case: $n = 1$, $\psi \in \Gamma \cup$ AXIOMS:

1.	ψ	$\psi \in \Gamma \cup$ AXIOMS
2.	$\psi \rightarrow (\varphi \rightarrow \psi)$	AX0
3.	$\varphi \rightarrow \psi$	M.P., 1,2

Inductive case: Assume true for all proofs of length less than n.

$$\alpha_1\ ,\ \alpha_2\ ,\ \ldots\ ,\ \alpha_{n-1}\ ,\ \psi$$

is a proof from $\Gamma\ \cup\ \{\varphi\}$.

By induction, $\Gamma \vdash \varphi \to \alpha_i$, $1 \leq i \leq n-1$.

ψ follows from α_i, α_j by M.P., $i, j < n$

$$\alpha_j \quad = \quad (\alpha_i \to \psi)$$

1.	$\varphi \to \alpha_i$	$\Gamma \vdash \varphi \to \alpha_i$
2.	$\varphi \to (\alpha_i \to \psi)$	$\Gamma \vdash \varphi \to \alpha_j$
3.	$(\varphi \to \alpha_i) \quad \to$ $((\varphi \to (\alpha_i \to \psi)) \to (\varphi \to \psi))$	AX0
4.	$(\varphi \to (\alpha_i \to \psi)) \to (\varphi \to \psi)$	M.P., 1,3
5.	$\varphi \to \psi$	M.P., 2, 4

♠

Proof by Contradiction MetaTheorem $=$ $\neg$ **Intro**

If $\Gamma \cup \{\varphi\} \vdash \bot$

Then $\Gamma \vdash \neg\varphi$

Proof: Suppose $\Gamma \cup \{\varphi\} \vdash \bot$

By the Deduction MetaTheorem, $\Gamma \vdash (\varphi \to \bot)$

1.	$\varphi \to \bot$	
2.	$(\varphi \to \bot) \to \neg\varphi$	AX0
3.	$\neg\varphi$	M.P. 1,2

♠

Exercise: If $\Gamma \cup \{\neg\varphi\} \vdash \bot$

Then $\Gamma \vdash \varphi$.

Change Bound Variable Lemma

If y does not occur freely in φ and if y is substitutable for x in φ, then

$$\vdash \forall x(\varphi) \;\leftrightarrow\; \forall y(\varphi[x \leftarrow y])$$

Proof:

Show $\forall x(\varphi) \vdash \varphi[x \leftarrow y]$

1.	$\forall x(\varphi)$	$\vdash$	$\forall x(\varphi)$	Assumption
2.	$\forall x(\varphi)$	$\vdash$	$\forall x(\varphi) \rightarrow \varphi[x \leftarrow y]$	AX2
3.	$\forall x(\varphi)$	$\vdash$	$\varphi[x \leftarrow y]$	M.P., 1,2
4.	$\forall x(\varphi)$	$\vdash$	$\forall y(\varphi[x \leftarrow y])$	$\forall$ Intro, 3
5.	$\vdash$		$\forall x(\varphi) \rightarrow \forall y(\varphi[x \leftarrow y])$	$\rightarrow$ Intro, 4

The converse is similar. ♠

Add a Constant MetaTheorem $= \exists$ Elim

If $\Gamma \vdash \exists x(\varphi)$ and $\Gamma, \varphi[x \leftarrow c] \vdash \psi$

where c does not occur in Γ, φ, or ψ.

Then $\Gamma \vdash \psi$.

Proof:

$\Gamma, \varphi[x \leftarrow c]$	$\vdash$	ψ	
Γ	$\vdash$	$\varphi[x \leftarrow c] \rightarrow \psi$	$\rightarrow$ Intro
Γ	$\vdash$	$\neg\psi \rightarrow \neg\varphi[x \leftarrow c]$	AX0, MP
$\Gamma, \neg\psi$	$\vdash$	$\neg\varphi[x \leftarrow c]$	
$\Gamma, \neg\psi$	$\vdash$	$\neg\varphi[x \leftarrow z]$	z new, Lemma **??**
$\Gamma, \neg\psi$	$\vdash$	$\forall z(\neg\varphi[x \leftarrow z])$	$\forall$ Intro
$\Gamma, \neg\psi$	$\vdash$	$\forall x(\neg\varphi)$	Change Bound Variable
$\Gamma, \neg\psi$	$\vdash$	$\neg\forall x(\neg\varphi)$	given
$\Gamma, \neg\psi$	$\vdash$	$\perp$	
Γ	$\vdash$	ψ	Proof by Contradiction

♠

Lemma 13.13 *If $\Delta \vdash \alpha[x \leftarrow c]$ where neither z nor c occurs in Δ or α, then $\Delta \vdash \alpha[x \leftarrow z]$*

Proof: By induction on length of proof of $\Delta \vdash \alpha[x \leftarrow c]$.
♠

Proposition 13.14 $\vdash \forall xyz((x = y \wedge y = z) \to x = z)$

Proof:

1.	$\{(x = y \wedge y = z)\} \vdash x = y$	$\wedge$ Elim
2.	$\{(x = y \wedge y = z)\} \vdash y = z$	$\wedge$ Elim
3.	$\vdash x = y \to y = x$	Lect. 13, sl. 12
4.	$\{(x = y \wedge y = z)\} \vdash y = x$	MP 1, 3
5.	$\vdash z = z$	AX 1a
6.	$\{(x = y \wedge y = z)\} \vdash (y = x \wedge z = z)$	$\wedge$ Intro 4, 5
7.	$\vdash (y = x \wedge z = z) \to (y = z \to x = z)$	AX 1c
8.	$\{(x = y \wedge y = z)\} \vdash (y = z \to x = z)$	MP 6, 7
9.	$\{(x = y \wedge y = z)\} \vdash x = z$	MP 2, 8
10.	$\vdash (x = y \wedge y = z) \to x = z$	$\to$ Intro 9
11.	$\vdash \forall z((x = y \wedge y = z) \to x = z)$	$\forall$ Intro. 10
12.	$\vdash \forall y \forall z((x = y \wedge y = z) \to x = z)$	$\forall$ Intro. 11
13.	$\vdash \forall x \forall y \forall z((x = y \wedge y = z) \to x = z)$	$\forall$ Intro. 12