Vocabulary: $\Sigma = (\Phi, \Pi, r)$: function symbols, predicate symbols, arity function, "=" $\in \Pi$.

Defines a type of structure by defining what may be said about it.

terms, atomic formulas, formulas: $\mathcal{L}(\Sigma)$

The statements that may be made about such a structure using boolean operators and quantification over variables.

Structure of vocabulary Σ, $\mathcal{A} = (U, \mu) \in \mathrm{STRUC}[\Sigma]$

A set of data that defines the meaning of the structure's objects, constants, relations, and functions, so that every statement in $\mathcal{L}(\Sigma)$ *becomes meaningful.*

Tarski's Definition of Truth:

By induction on the definition of formulas, we define what it means for "a structure to satisfy a formula", or equivalently for "a formula to be true in a structure".

$$
\begin{aligned}
(|\mathcal{A}|, \mu) \models t_1 = t_2 &\Leftrightarrow \mu(t_1) = \mu(t_2) \\
(|\mathcal{A}|, \mu) \models R_j(t_1, \ldots, t_{r(R_j)}) &\Leftrightarrow \langle \mu(t_1), \ldots, \mu(t_{r(R_j)}) \rangle \in R_j^{\mathcal{A}} \\
(|\mathcal{A}|, \mu) \models \neg\varphi &\Leftrightarrow (|\mathcal{A}|, \mu) \not\models \varphi \\
(|\mathcal{A}|, \mu) \models \varphi \vee \psi &\Leftrightarrow (|\mathcal{A}|, \mu) \models \varphi \text{ or } (|\mathcal{A}|, \mu) \models \psi \\
(|\mathcal{A}|, \mu) \models (\forall x)\varphi &\Leftrightarrow (\text{for all } a \in |\mathcal{A}|) \\
&\qquad (|\mathcal{A}|, \mu, a/x) \models \varphi
\end{aligned}
$$

Play Tarski's Truth Game!!!

world: $\mathcal{W}$; sentence: φ; players: A, B

A asserts that $\mathcal{W} \models \varphi$; B denies that $\mathcal{W} \models \varphi$.

The game rules depend inductively on the formula φ:

φ is atomic: A wins iff $\mathcal{W} \models \varphi$.

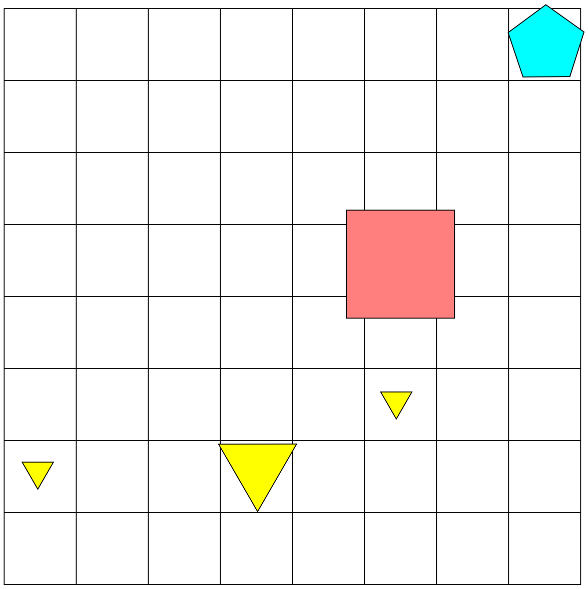
$\varphi \equiv \alpha \vee \beta$: A asserts $\mathcal{W} \models \alpha$ or A asserts $\mathcal{W} \models \beta$.

$\varphi \equiv \alpha \wedge \beta$: B denies $\mathcal{W} \models \alpha$ or B denies $\mathcal{W} \models \beta$.

$\varphi \equiv \neg\alpha$: A and B switch rôles, and B asserts $\mathcal{W} \models \alpha$.

$\varphi \equiv \exists x(\psi)$: A chooses an element from $|\mathcal{W}|$, assigning it a name n. A asserts that $\mathcal{W}' \models \psi[x \leftarrow n]$.

$\varphi \equiv \forall x(\psi)$: B chooses an element from $|\mathcal{W}|$, assigning it a name n. B denies that $\mathcal{W}' \models \psi[x \leftarrow n]$.



A asserts: $\forall x \exists y(\text{Tet}(x) \rightarrow (\text{Larger}(y, x) \wedge \text{RightOf}(y, x)))$

A asserts: $\forall x \exists y(\neg\text{Tet}(x) \vee (\text{Larger}(y, x) \wedge \text{RightOf}(y, x)))$

B chooses: $x := n_1$

A asserts: $\exists y(\neg\text{Tet}(n_1) \vee (\text{Larger}(y, n_1) \wedge \text{RightOf}(y, n_1)))$

A chooses: $y := n_2$

A asserts: $\neg\text{Tet}(n_1) \vee (\text{Larger}(n_2, n_1) \wedge \text{RightOf}(n_2, n_1))$

A asserts: $(\text{Larger}(n_2, n_1) \wedge \text{RightOf}(n_2, n_1))$

B chooses: **(**$either$)

A wins

Theorem 12.1 *For any vocabulary* Σ*,* $\mathcal{W} \in \mathrm{STRUC}[\Sigma]$*,* $\varphi \in \mathcal{L}(\Sigma)$*,*

in the game where A *asserts and* B *denies that* $\mathcal{W} \models \varphi$*,*

$$\mathcal{W} \models \varphi \ \Leftrightarrow \ A \textit{ has a winning strategy}$$

$$\mathcal{W} \not\models \varphi \ \Leftrightarrow \ B \textit{ has a winning strategy}$$

Proof: Think about this! You may be asked to check the details on a future homework. ♠

The Tarski Truth game is probably the best way to think about the meaning of first-order formulas. In particular, note that the order of quantifiers corresponds to the order that choices are made in the game.

Example: Structure is a Binary String

$$\begin{aligned}\Sigma_s &= (\emptyset, \{<, S\}, \{\langle <, 2\rangle, \langle S, 1\rangle\}) \\ &= (; <^2, S^1)\end{aligned}$$

$$w \quad = \quad 01101$$

$$\mathcal{A}_w \quad = \quad \langle \{0, 1, \ldots, 4\}, <, \{1, 2, 4\}\rangle \ \in \ \text{STRUC}[\Sigma_s]$$

$$\alpha \ \equiv \ (\exists x)(\forall y)(y \leq x \ \wedge \ S(x))$$

$$\beta \ \equiv \ (\forall xy)((x < y \wedge \neg S(x) \wedge \neg S(y)) \rightarrow (\exists z)(x < z < y))$$

$$\mathcal{A}_w \ \models \ \alpha \ \wedge \ \beta$$

$$\Sigma_{gen} \quad = \quad (; F^1, P^2, S^2)$$

$$\mathcal{B}_0 = \langle U_0, F_0, P_0, S_0 \rangle \in \text{STRUC}[\Sigma_{gen}]$$

$$U_0 = \{\text{Abraham, Isaac, Rebekah, Sarah, } \ldots\}$$

$$F_0 = \{\text{Sarah, Rebekah, } \ldots\}$$

$$P_0 = \{\langle\text{Abraham,Isaac}\rangle, \langle\text{Sarah,Isaac}\rangle, \ldots\}$$

$$S_0 = \{\langle\text{Abraham,Sarah}\rangle, \langle\text{Isaac,Rebekah}\rangle, \ldots\}$$

$$\varphi_{sibling}(x, y) \equiv \exists fm(x \neq y \wedge f \neq m \wedge P(f, x) \wedge P(f, y) \wedge P(m, x) \wedge P(m, y))$$

$$\varphi_{aunt}(x, y) \equiv \exists ps(F(x) \wedge P(p, y) \wedge \varphi_{sibling}(p, s) \wedge (s = x \vee S(x, s)))$$

Example: Models of Number Theory

$$\mathbf{N} \quad = \quad (\mathbf{N}, 0, \sigma, +, \times, \uparrow, <)$$

N is the "standard model of the natural numbers". But we can define other models where the statements of number theory are meaningful!

Let p be a prime number.

$$\mathbf{Z}/p\mathbf{Z} \quad = \quad (\{0, 1, \ldots, p-1\}, 0, +1_p, +_p, \times_p, \uparrow_p, \emptyset)$$

$$\mathbf{N},\ \mathbf{Z}/p\mathbf{Z} \ \in \ \text{STRUC}[\Sigma_N]$$

$$\text{MultInverses} \quad \equiv \quad (\forall u)(u = 0 \vee (\exists v)(u \times v = 1))$$

$$\mathbf{Z}/p\mathbf{Z} \ \models \ \text{MultInverses}$$

$$\mathbf{N} \ \models \ \neg\text{MultInverses}$$

The Tarski Game in Z/3Z

A asserts: $\mathbf{Z}/3\mathbf{Z} \models \forall u(u = 0 \vee (\exists v)(u \times v = 1))$

B chooses: u from $\{0, 1, 2\}$

A asserts: $\mathbf{Z}/3\mathbf{Z} \models (\square = 0 \vee (\exists v)(\square \times v = 1))$

A chooses:

A wins by choosing "$u = 0$" if this is true, or by making v the real inverse of B's number u otherwise.

Tarski Truth Works for "AND"

Proposition 12.2

$$(|\mathcal{A}|, \mu) \models \varphi \wedge \psi \quad \Leftrightarrow \quad (|\mathcal{A}|, \mu) \models \varphi \text{ *and* } (|\mathcal{A}|, \mu) \models \psi$$

Proof:

$$
\begin{aligned}
& (|\mathcal{A}|, \mu) \models \varphi \wedge \psi \\
\Leftrightarrow\ & (|\mathcal{A}|, \mu) \models \neg(\neg\varphi \vee \neg\psi) \\
\Leftrightarrow\ & \text{not } (|\mathcal{A}|, \mu) \models \neg\varphi \vee \neg\psi \\
\Leftrightarrow\ & \text{not } [((|\mathcal{A}|, \mu) \models \neg\varphi) \text{ or } ((|\mathcal{A}|, \mu) \models \neg\psi)] \\
\Leftrightarrow\ & (|\mathcal{A}|, \mu) \not\models \neg\varphi \text{ and } (|\mathcal{A}|, \mu) \not\models \neg\psi \\
\Leftrightarrow\ & (|\mathcal{A}|, \mu) \models \varphi \text{ and } (|\mathcal{A}|, \mu) \models \psi
\end{aligned}
$$

♠

Tarski Truth Works for "$\exists$"

Proposition 12.3

$(|\mathcal{A}|, \mu) \models (\exists x)\varphi \quad \Leftrightarrow \quad (\textit{exists } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \models \varphi$

Proof:

$$
\begin{aligned}
& (|\mathcal{A}|, \mu) \models (\exists x)\varphi \\
\Leftrightarrow\ & (|\mathcal{A}|, \mu) \models \neg(\forall x)\neg\varphi \\
\Leftrightarrow\ & (|\mathcal{A}|, \mu) \not\models (\forall x)\neg\varphi \\
\Leftrightarrow\ & \text{not (for all } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \models \neg\varphi \\
\Leftrightarrow\ & (\text{for some } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \not\models \neg\varphi \\
\Leftrightarrow\ & (\text{for some } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \models \varphi
\end{aligned}
$$

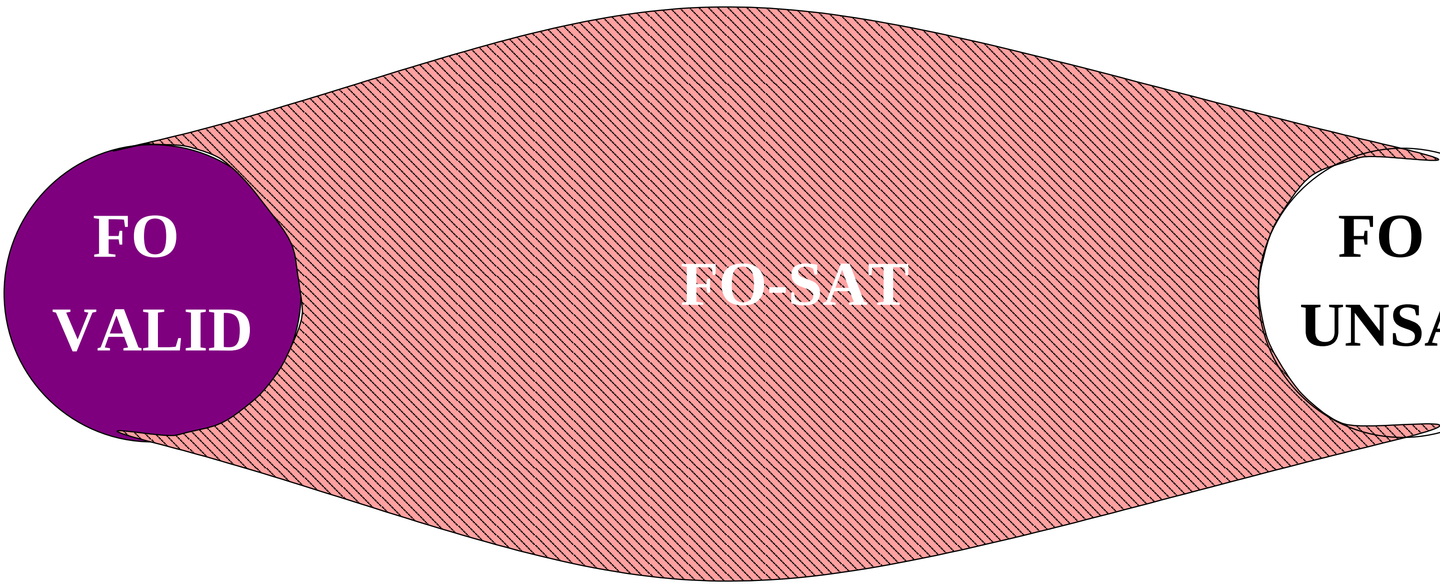
♠

Definition 12.4 A formula $\varphi \in \mathcal{L}(\Sigma)$ is *satisfiable* iff there exists $\mathcal{A} \in \mathrm{STRUC}[\Sigma]$, $\mathcal{A} \models \varphi$.

φ is *valid* ($\models \varphi$) iff for all $\mathcal{A} \in \mathrm{STRUC}[\Sigma]$, $\mathcal{A} \models \varphi$.

A set of formulas $\Gamma \subseteq \mathcal{L}(\Sigma)$ *semantically implies* a formula $\varphi \in \mathcal{L}(\Sigma)$ ($\Gamma \models \varphi$) iff for all $\mathcal{A} \in \mathrm{STRUC}[\Sigma]$,

$$\mathcal{A} \models \Gamma \quad \Rightarrow \quad \mathcal{A} \models \varphi$$ ♠



$$\text{FO-VALID} \quad = \quad \{\varphi \mid \models \varphi\}$$

"The FO-VALID formulas are the set of formulas φ such that the empty set of formulas $\models \varphi$, that is, such that any structure of the correct type models φ."

Note how we have overloaded the symbol "$\models$". It can refer to:

- A structure modeling a formula
- A formula being FO-valid, or
- A set of sentences semantically implying a formula

Proposition 12.5 *Let $f(\varphi) = \neg\varphi$. Then,*

f : *FO-VALID $\leq$ FO-UNSAT* *and*

f : *FO-UNSAT $\leq$ FO-VALID*

The key fact justifying these definitions is that semantic implication is the same as propositional implication:

Proposition 12.6

$$\{\psi\} \models \varphi \;\Leftrightarrow\; \models (\psi \to \varphi)$$

$$\Leftrightarrow\; \models (\neg\psi \;\vee\; \varphi)$$

Notation: For $\Gamma \subseteq \mathcal{L}(\Sigma)$, $\varphi \in \mathcal{L}(\Sigma)$, "$\Gamma \vdash \varphi$" is read, "$\Gamma$ proves φ", and means, "There is a first-order proof of φ assuming Γ."

We are currently dealing with *two different* proof systems for FO predicate calculus, the one in [P] and the (more familiar-looking) Fitch system used in [BE].

While Fitch has several proof rules, [P] gets by with only one. (On the other hand, [P] has lots of axioms which will take us the rest of this lecture to review.)

[P]'s only proof rule is:

Modus Ponens (M.P.): $\frac{\Gamma \vdash \varphi \to \psi, \ \Gamma \vdash \varphi}{\Gamma \vdash \psi}$

Proposition 12.7 *Modus Ponens preserves truth, validity, and semantic implication, i.e.,*

if $\mathcal{A} \models \varphi \rightarrow \psi$ *and* $\mathcal{A} \models \varphi$ **then** $\mathcal{A} \models \psi$.

if $\Gamma \models \varphi \rightarrow \psi$ *and* $\Gamma \models \varphi$ **then** $\Gamma \models \psi$.

Proof: Suppose $\Gamma \models \varphi$ and $\Gamma \models \varphi \rightarrow \psi$.

Let $\mathcal{A}$ be arbitrary such that $\mathcal{A} \models \Gamma$.

$\mathcal{A} \models \varphi$, $\mathcal{A} \models \neg\varphi \vee \psi$

$\mathcal{A} \models \psi$

$\Gamma \models \psi$ ♠

Generalizations

If φ is a first-order formula,

then $\forall x(\varphi)$ is called a **generalization** of φ.

Proposition 12.8 *If* $\models \varphi$, *then* $\models \forall x(\varphi)$.

Proof: Assume that $\models \varphi$ where $\varphi \in \mathcal{L}(\Sigma)$.

Let $\mathcal{A} \in \text{STRUC}[\Sigma]$ be arbitrary.

Let $a \in |\mathcal{A}|$ be arbitrary; $(\mathcal{A}, a/x) \in \text{STRUC}[\Sigma]$

$(\mathcal{A}, a/x) \models \varphi$

for all $a \in |\mathcal{A}|$, $(\mathcal{A}, a/x) \models \varphi$

$\mathcal{A} \models \forall x(\varphi)$

$\models \forall x(\varphi)$

♠

all generalizations of the following:

AX0: Tautologies on **at most three boolean variables,** with first-order formula substituted for the variables.

1. $x_1 \to x_1$
2. $x_1 \to (x_1 \vee x_2)$
3. $x_1 \vee \neg x_1$
4. $x_1 \to (\neg x_1 \to x_2)$

1. $(\forall u)(\exists v)E(u,v) \ \to \ (\forall u)(\exists v)E(u,v)$
2. $(\forall z)(z < z + z) \to ((\forall z)(z < z + z) \vee (\forall y)(y < z))$
3. $(\exists z)R(z) \ \vee \ \neg(\exists z)R(z)$
4. $\text{prime}(17) \to (\neg\text{prime}(17) \to 0 \neq 0)$

Proposition 12.9 *All members of AX0 are valid.*

Equality Axioms

all generalizations of the following:

AX1a $t = t$, for any term t

AX1b $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow f(t_1, \ldots, t_k) = f(t'_1, \ldots, t'_k)$ for terms $t_1, \ldots, t'_k$, $f \in \Phi$, $r(f) = k$

AX1c $(t_1 = t'_1 \wedge \cdots \wedge t_k = t'_k) \rightarrow (R(t_1, \ldots, t_k) \rightarrow R(t'_1, \ldots, t'_k))$ for terms $t_1, \ldots, t'_k$, $R \in \Pi$, $r(R) = k$

Proposition 12.10 *Every instance of AX1 is valid.*

Proof: Because "=" is interpreted as "identically equal".
♠

Definition 12.11 Term t is *substitutable* for variable x in φ iff no free occurrence of x in φ is within the scope of a quantifier for a variable z occurring in t.

$\varphi[x \leftarrow t]$ is the result of substituting t for all free occurrences of x in φ.

We never use this expression unless t is substitutable for x in φ. ♠

$$\begin{aligned} \alpha &\equiv (\exists y)(y < x) \\ \alpha[x \leftarrow z+1] &\equiv (\exists y)(y < z+1) \\ \alpha[x \leftarrow f(u)+v] &\equiv (\exists y)(y < f(u)+v) \\ \alpha' &\equiv (\exists y)(y < y) \end{aligned}$$

$z+1$, u, $f(u)$, $f(u)+v$ are substitutable for x in α.

y, $y+1$ are not substitutable for x in α or φ.

$$\begin{aligned} \alpha &\equiv \text{“}x \text{ is not the least element”} \\ \alpha[x \leftarrow z+1] &\equiv \text{“}z+1 \text{ is not the least element”} \end{aligned}$$

Instantiation Axioms:

all generalizations of the following

AX2: $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t]$, x a variable, t a term, t substitutable for x in φ.

Proposition 12.12 *Every instance of AX2 is valid.*

Proof: Let $\forall x(\varphi) \rightarrow \varphi[x \leftarrow t] \in$ AX2.

If $\forall x(\varphi)$ is false in the current interpretation $\mathcal{A}$, we are done. Otherwise, by the definition of Tarski Truth for $\forall$, $\varphi[x \leftarrow a]$ is true for any element a of $\mathcal{A}$, so it's true for the element represented by t. (And truth is preserved by substituting equals.)

♠

Re-Labeling Bound Variables

Lemma 12.13 *Let $\mathcal{A}, \mathcal{A}'$ be identical except for how they interpret some variables not free in φ. Then,*

$$\mathcal{A} \models \varphi \qquad \Leftrightarrow \qquad \mathcal{A}' \models \varphi$$

Proof: By induction on φ.

Base case: $\varphi \equiv R(t_1, \ldots, t_k)$

Inductive case 1: $\varphi \equiv \neg\psi$

Inductive case 2: $\varphi \equiv (\alpha \vee \beta)$

Inductive case 2: $\varphi \equiv \forall x(\psi)$

♠

Generalization Axioms

all generalizations of the following:

AX3: $\varphi \to \forall x(\varphi)$, where x does not occur freely in φ.

Proposition 12.14 *Every instance of AX3 is valid.*

Proof: Let $\varphi \to \forall x(\varphi) \in$ AX3.
Let $\mathcal{A} \in$ STRUC$[\Sigma]$ be arbitrary.
Suppose $\mathcal{A} \models \varphi$

$$(|\mathcal{A}|, \mu) \models \forall x(\varphi) \quad \Leftrightarrow \quad (\text{for all } a \in |\mathcal{A}|)(|\mathcal{A}|, \mu, a/x) \models \varphi$$

By Lemma 12.13, $\mathcal{A} \models \forall x(\varphi)$

♠

But didn't we just use Generalization to prove Generalization?

Not quite. We used mathematical generalization in the real world to prove that generalization in this logical formalism preserves truth.

One Last Set of Axioms

all generalizations of the following:

AX4: $\forall x(\varphi \to \psi) \ \to \ (\forall x(\varphi) \to \forall x(\psi))$

Proposition 12.15 *Every instance of AX4 is valid.*

Proof:

$$(\forall x)(\varphi \to \psi) \ \to \ ((\forall x)(\varphi) \to (\forall x)(\psi)) \quad \in \quad \text{AX4}$$

Suppose $\mathcal{A} \models \forall x(\varphi \to \psi)$. (...finish on the whiteboard...)

♠