CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #7: Graph Coloring Theorems
(Tucker Section 2.4)

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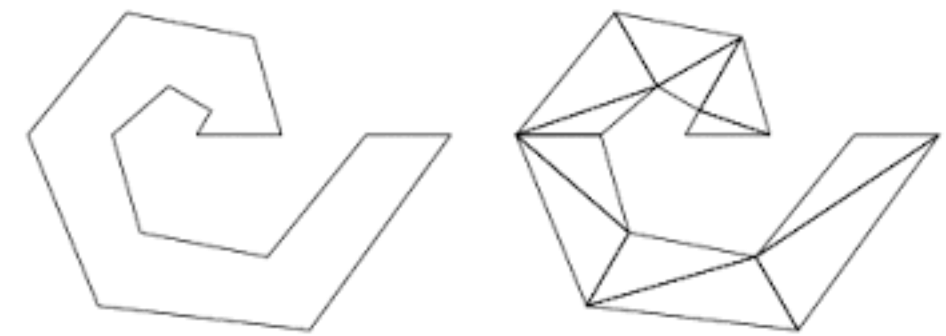
Graph Coloring Theorems

- Coloring Triangulations of Polygon
- The Art Gallery Problem
- Chromatic Number and Degree
- Vizing's Edge Coloring Theorem
- 5-Coloring a Planar Graph
- Chromatic Polynomials
- Calculating Chromatic Polynomials

Coloring Triangulations



- A **triangulation** of a polygon is a division of it into triangles by drawing **chords** from vertex to vertex inside it. This results in a planar graph.
- It's easy to show by induction, or by Euler's formula, that an n-gon yields exactly n-2 triangles from this process.
- The resulting graph also always has a chromatic number of 3.

Coloring Triangulations

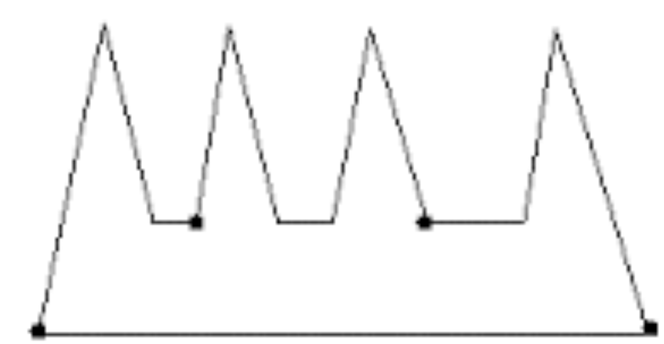
- Let's prove this by induction on n, the number of sides is the polygon. The base is n = 3.
- The IH is that the triangulation of any k-gon for $k < n$ can be 3-colored.
- Given an n-gon, any chord we draw gives us an i-gon and a j-gon, where $i + j = n + 2$, since we have created two new sides. We can 3-color both of these by the IH.
- But they intersect in only two vertices, and we can choose colors to match for those.

The Art Gallery Problem

- In the Art Gallery Problem, we have a polygonal gallery and we want to station guards at vertices, as few as possible, to have every point in the gallery in view of a guard. It suffices for them to see all other vertices.
- We show that floor(n/3) guards suffice for n walls. Just triangulate the graph and 3-color the result. One color has at most floor(n/3) vertices, and we put guards at all vertices of that color.

The Art Gallery Problem

- Since every triangle in the graph has one vertex of each color, our guards can see all of every triangle, and thus see the entire graph.
- In fact floor(n/3) guards are necessary. The graph in this picture is an example of a family of polygons with n "teeth" and 3n+1 edges. A gallery of this shape requires at least floor((3n+1)/3) = n guards because no guard can see more than one of the n tooth points.



n = 13 $\lfloor n/3 \rfloor$ = 4

Chromatic Number and Degree

- A theorem of Brooks says that the degree of graph is an upper bound on the chromatic number, with two exceptions: odd cycles and K_n.
- This is not at all a tight bound, as the graph W_n is 3- or 4-colorable but has degree n-1.
- The size of largest included K_n in a graph is certainly a lower bound on the chromatic number, but there are triangle-free graphs of arbitrarily high chromatic number (see Exercise 18 of section 2.4, which produces huge graphs).

Vizing's Edge Coloring Theorem

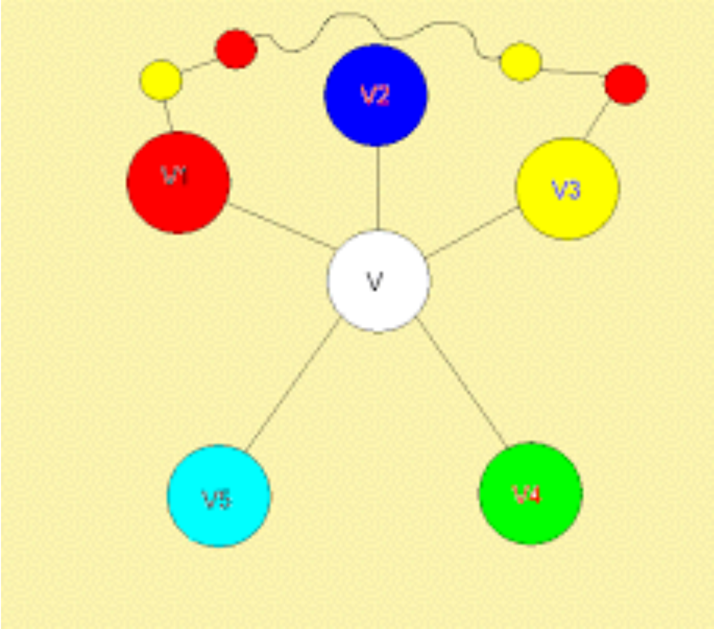
- The phone company used to implement huge multigraphs with colored wires. You didn't want two wires of the same color coming out of the same point, and you only had so many colors. You'd like an edge coloring.
- If a graph has degree d, any edge coloring needs at least d colors.
- Vizing proved the there is always an edge coloring with d+1 colors, so that the **edge chromatic number** is always d or d+1.

5-Coloring a Planar Graph

- We won't prove the 4-color theorem here, but we can show that any planar graph can be 5-colored. The proof is by induction on the number of vertices, n.
- Recall that planar graph must have vertex of degree at most 5. Remove that vertex v and use the IH to color the remaining graph.
- If the neighbors of v don't have five different colors, we are done because there is a color left for v. So we look at the other case.

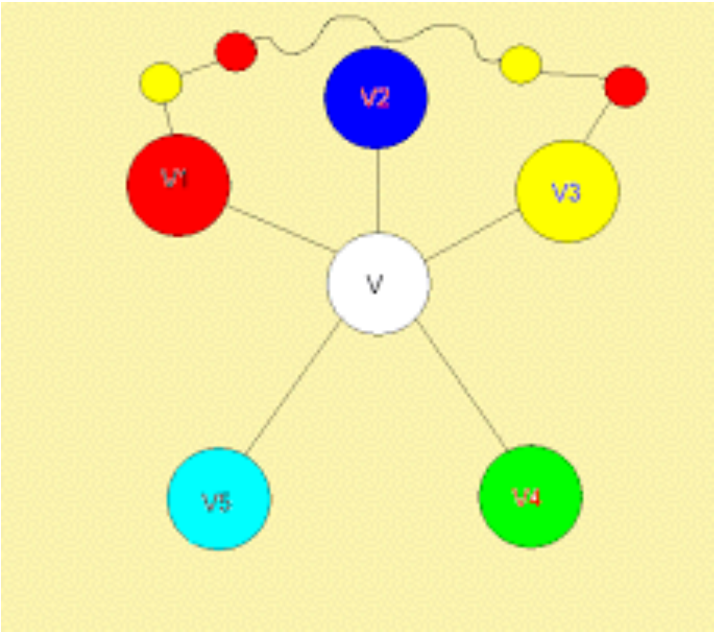
5-Coloring a Planar Graph

- We can draw the five neighbors of v as in this picture. Look at the regions of the graph that we can get to from the red or yellow neighbors, using only red and yellow nodes.
- We might be able to get from the red to the yellow neighbor that way. But if we can do that, we can't get from the dark blue to the green neighbor using only *those* two colors.



5-Coloring a Planar Graph

- If we can't get from the red neighbor to the yellow neighbor using only red and yellow, switch red and yellow in the region that you *can* get to. Then we can color v red, and we win.
- Otherwise, switch dark blue and green in the region of those two colors containing the dark blue neighbor, then color v dark blue.
- The coloring is still legal.



Chromatic Polynomial

- Let G be any graph, and define $P_k(G)$ to be the number of colorings of G with k colors. For example, if G has no edges, $P_k(G) = k^n$, where n is the number of vertices in G.

- It turns out that this function is always a polynomial in k, and thus it is called the chromatic polynomial. (We'll prove this soon.)

- Examples: If G is a path, or in fact any tree, then $P_k(G) = k(k-1)^{n-1}$. $P_k(K_n)$ is the number P(k, n) of no-repeat sequences of length n from a k-element set, or $k(k-1)\ldots(k-n+1)$.

Calculating Chromatic Polys

- Let C_4 be the cycle a-b-c-d. There are $k(k-1)^2$ colorings with a and c the same color, and $k(k-1)(k-2)^2$ with them different. This gives $k(k-1)(k^2 - 3k + 3)$ in all.
- Theorem 6 in Tucker lets us compute $P_k(G)$ from the polynomials of two other graphs. If x and y are two non-adjacent vertices in G, then $P_k(G) = P_k(G_{+xy}) + P(G_{x=y})$. Here G_{+xy} is the graph obtained by adding (x, y) to G, and $G_{x=y}$ is the graph obtained by merging x and y.

Calculating Chromatic Polys

- Similarly, $P_k(G) = P_k(G_{-xy}) - P_k(G_{x=y})$ by simple arithmetic on the first result.
- These facts let us compute $P_k(G)$ for any graph G, by reducing to the cases of smaller n and of graphs that are either empty or complete. Remember that if a graph is disconnected, we just multiply together the polynomials of its components.

Chromatic Poly Example

- Let's return to the case of $G = C_4$, the four-node cycle. Let x and y be adjacent nodes. Then G_{-xy} is a 4-node tree, with $P(G_{-xy}) = k(k-1)^3$. And $G_{x=y} = C_3$, so $P(G_{x=y}) = k(k-1)(k-2)$. (Note that $C_3 = K_3$.)
- This gives us $P(G) = k(k-1)[(k-1)^2 - (k-2)] = k(k-1)(k^2 - 3k + 3)$ as we computed before.
- And of course by induction, P(G) is always a polynomial in k.