CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #5: Euler and Hamilton Circuits
(Tucker Section 2.1, 2.2)
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Euler and Hamilton Circuits

- Multigraphs, Trails, and Cycles
- The Königsberg Bridge Problem
- Euler Paths and Cycles
- Proofs of the Euler Cycle Theorem
- Hamilton Paths and Circuits
- Ruling Out Hamilton Circuits
- Hamilton Paths in Tournaments

Multigraphs

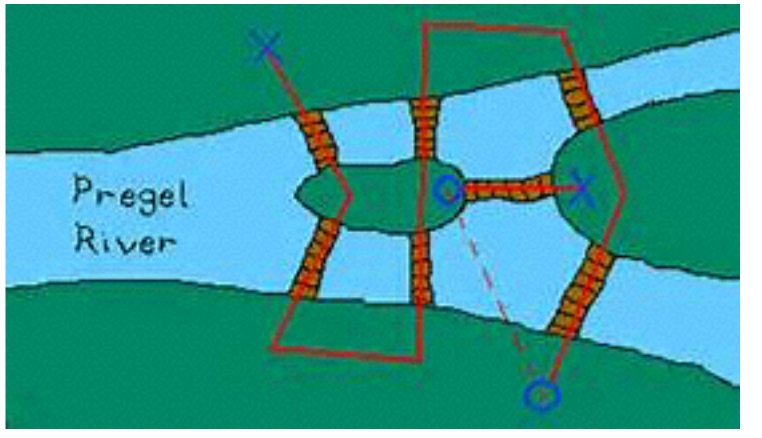
- We've defined graphs and directed graphs to forbid parallel edges.
- But there are many situations where there are multiple ways to go from x to y in one step, and it matters which we take.
- A **multigraph** is a vertex set together with a set of undirected edges, where more than one edge might connect the same pair of vertices. A **directed multigraph** is similar.

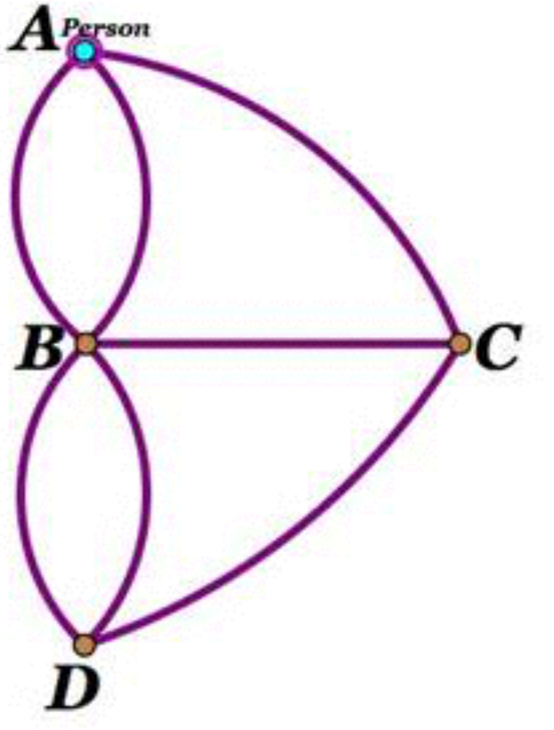
Trails and Cycles

- A **trail** is a sequence of edges in a multigraph, where each edge connects to the next one. The same edge may not be used more than once, but parallel edges may be used.
- A **cycle** is a nonempty trail that begins and ends on the same vertex. In an ordinary undirected graph, a cycle is a path of three or more edges (none repeated) that goes from a vertex to itself. (Thus trees have no cycles.)

The Königsberg Bridges

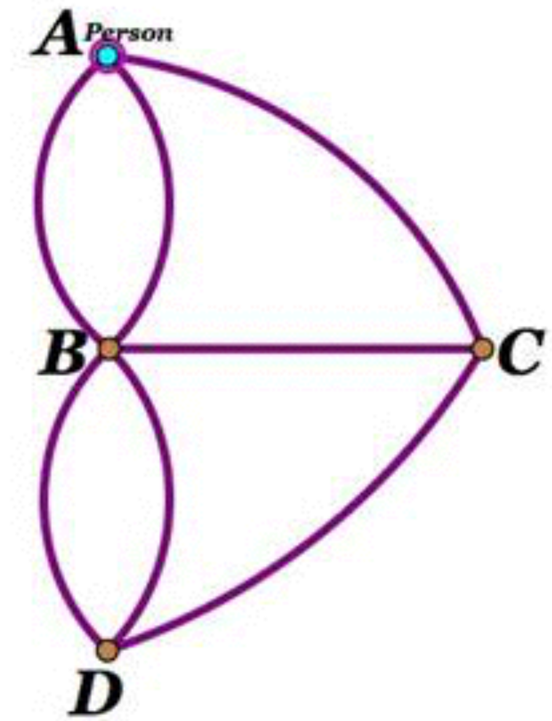
- The city of Königsberg had two islands and seven bridges as shown. The citizens wondered whether it was possible to take a single walk using each bridge exactly once.
- Fortunately the great Leonhard Euler lived in the city, and could tell them this was impossible.





Euler Paths and Cycles

- An **Euler path** in a multigraph is a trail that uses all the edges. If we added another bridge from B to C, we could have an Euler path going A-B-C-B-D-C-A-B-D. Note the two A-B, two B-C, and two B-D steps.



- An Euler cycle is an Euler path that returns to its starting point. If we added another bridge from A to D, we could extend the path above to a cycle.

Proving Euler’s Cycle Theorem

- Euler’s great insight was that the possibility of a path of cycle depended only on what we would call the multigraph of the bridges. The key to the solution is the degrees of the nodes.
- In any Euler cycle, every node is entered exactly the same number of times it is left. Therefore the number of edges used, which must be the degree of the node, is even.
- In an Euler path from x to y, x and y have odd degree and all other nodes have even degree.

Proving Euler's Cycle Theorem

- But that's only half of the proof. We have only shown that if an Euler path or cycle exists, then the degree condition must hold.
- How do we construct a path if the condition holds? By induction on the number of edges in the multigraph.
- The base case is one node with no edges, for which we will allow the trivial cycle.
- Suppose we have n edges, and all connected multigraphs with fewer satisfy the theorem.

Proving Euler’s Cycle Theorem

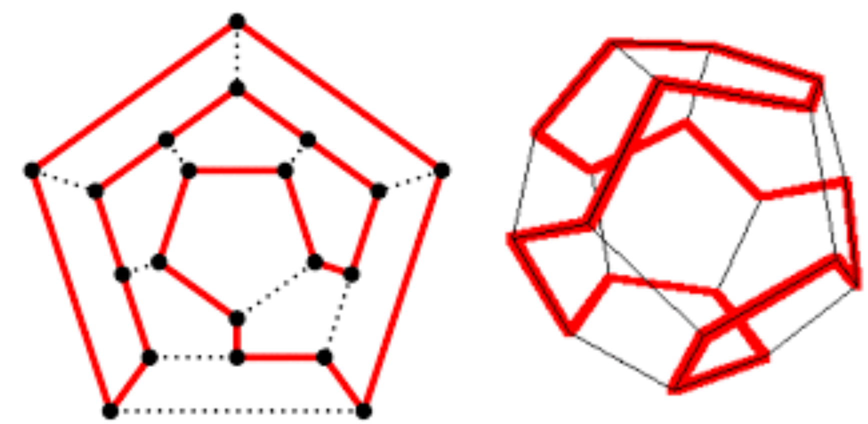
- Assume all nodes have even degree. Start at any node. Take an edge. Take a following edge. (The new node now has an odd number of edges left, so there is one.)
- Keep going until you return to the start point
- You have to, because every new node you reach except the start point has an odd number of edges left, so you can keep going, and you can’t keep going forever.

Proving Euler's Cycle Theorem

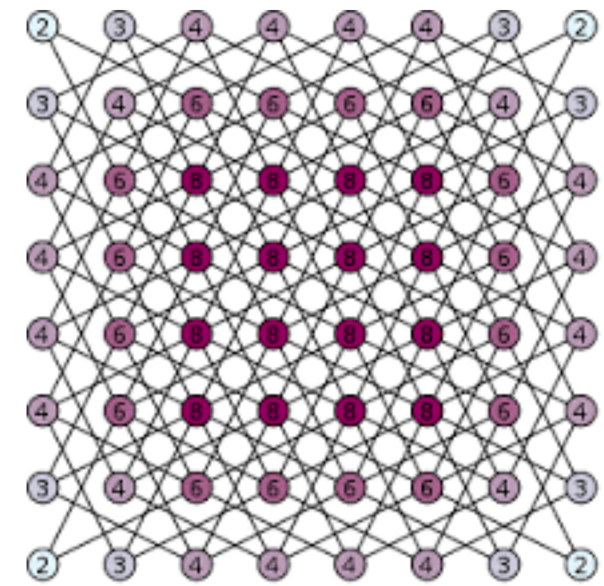
- Now you have a cycle, plus perhaps some remaining edges. Each node in the graph of remaining edges has even degree.
- This graph may not be connected, but by our IH each component of it must have an Euler path. And our cycle must reach each component because the first graph was connected. We just stitch all these cycles together.
- If we start with x and y of odd degree, start the first cycle at x and it can only end at y.

Hamilton Paths and Circuits

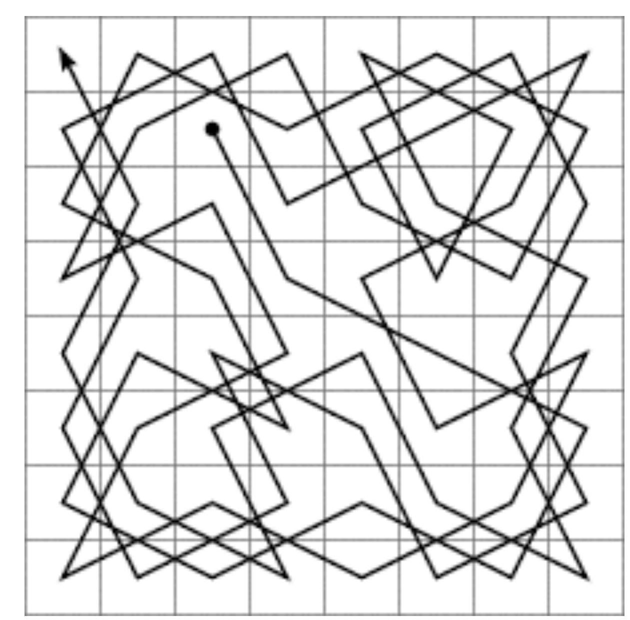
- In the 19th century William Hamilton marketed a puzzle where you were asked to find a tour of a dodecahedron, visiting each vertex exactly once. This is in fact possible.



Hamilton Paths and Circuits



- The knight's tour problem asks whether a chess knight can visit all 64 squares in 64 moves, returning to the start.
- This asks for a **Hamilton circuit** of the knight-move graph.
- A **Hamilton path** need not return to its start point.



Testing Hamiltonicity

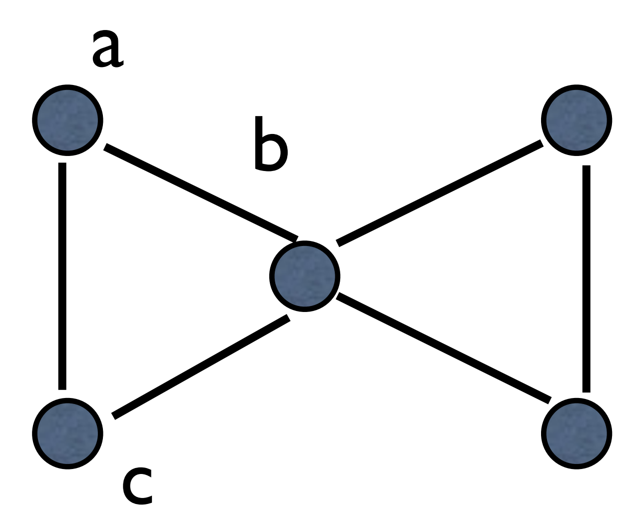
- It turns out that given a graph with n vertices, there is no known way to decide in a time polynomial in n whether there is a Hamilton path or circuit.
- The problem is NP-complete, because given any instance of the satisfiability problem, you can cook up a graph that has a Hamilton path if and only if the given formula is satisfiable. (If you haven't taken 311 or 501, we'll explain more about this later in the course.)

Testing Hamiltonicity

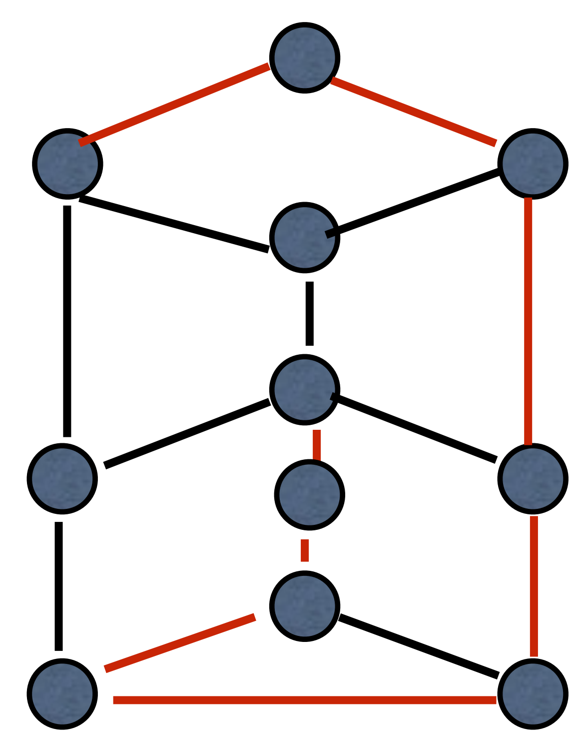
- In particular, there is presumably no easy condition, like the one for Euler circuits, that tells you whether a Hamilton path exists.
- But we can use our same general method to find this out on reasonably small graphs. We try to construct a path, keeping track of all choices that we make while doing so.
- If we succeed, then a path exists. If every choice leads to failure, it doesn't.

Ruling Out Hamilton Circuits

- If a node has degree 2, any Hamilton cycle must enter it on one edge and leave it on the other. So in the top graph, any tour must contain a-b-c or c-b-a, and is stuck.



- Tucker gives a more complicated argument for the bottom graph. The red edges are forced to be in any cycle.



Theorems About H-Circuits

- If a graph has n vertices and every vertex has degree $\geq n/2$, it has a Hamilton circuit.
- If G has a circuit H, and we draw G in the plane, let r_i be the number of regions of degree r inside the circuit, and r_i' be the number of such regions outside. Then $\Sigma_i (i - 2)(r_i - r_i') = 0$. Tucker uses this to show that a graph with three 4-regions and six 6-regions cannot have a Hamilton circuit.

Hamilton Paths in Tournaments

- Here's a theorem we can prove, involving Hamilton paths in **directed** graphs.
- A tournament on n nodes is a directed graph obtained by assigning a direction to every undirected edge in a K_n. This can encode the results of a round-robin tournament among n teams, where games cannot end in ties.
- We'll show that every tournament has a directed Hamilton path.

Hamilton Paths in Tournaments

- We use induction on n. If $n = 1$, the trivial path is a Hamilton path. If $n = 2$, the path consisting of the single edge is Hamilton.
- A tournament on n+1 nodes can be obtained by taking a tournament on n nodes, adding a new node, and adding an edge from the new node to or from each other vertex.
- Assume we have such a graph and let $P = v_1$-v_2-…-v_n be the Hamilton path on the n-node tournament given by the IH.

Hamilton Paths in Tournaments

- Let x be the new node. If the edge between x and v_1 goes from x, we win, because we can start a path with that edge and follow with P.
- Similarly if the edge between x and v_n goes to x, we win with P followed by that edge.
- If neither of these happen, there must be some vertex v_i such that the directed edges from v_i to x and from x to v_{i+1} are in the graph. We can splice x into P using these two edges, removing the edge from v_i to v_{i+1}.