COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #35: Conway's Number System
(from Conway, *On Numbers and Games*
and Berlekamp, Conway, and Guy, *Winning Ways*)
David Mix Barrington
12 December 2016

Conway’s Number System

- Review: Games and Numbers
- Single-Stalk Hackenbush
- What are Real Numbers?
- Infinitesimals
- Ordinals
- The Games “Up” and “Down”
- Multiplying Numbers

Conway’s Combinatorial Games

- Conway recursively defines a **game** to be a set of **left options** and a set of **right options**, each of which is a game.
- The base case of the recursion is the **zero game** with no options for either player (so that the second player wins).
- Any game must end in a finite (but possibly unbounded) number of moves, even if it has infinitely many states.

The Order on Games

- Given any game, there is a winner under optimal play if Left moves first, and a winner under optimal play if Right moves first.
- A game where Left wins in both scenarios is called **positive**, and one where Right wins both is called **negative**. One where the first player wins is called **fuzzy**, and one where the second player wins is called a **zero game**.
- **Non-partisan** games, where both players have the same options, are always zero or fuzzy.

The Order on Games

- We define a partial order on games denoted by the usual symbols >, ≥, =, ≤, and <.
- Any game G has an **additive inverse** -G made by switching the roles of Left and Right. Using the **game sum** operation, G + (-G) is always a zero game.
- Given two games G and H, we say that G > H if G - H is positive, and that G < H if G - H is negative. If G - H is a zero game, we say G = H, and if G - H if fuzzy, G and H are called **confused**.

Some Games are Numbers

- We proved the Sprague-Grundy Theorem, that any finite impartial game is equal to a single-pile Nim game. The value of such a game is called a **nimber**, written *k for a pile of k stones.
- Some partisan games have values called **numbers**, and today we will look at the resulting system of numbers (called **surreal numbers**).

Some Games are Numbers

- For a game G to be a number, each of its left and right options (if any) must be a number, and every left option must be less than every right option according to the order we defined.
- It will follow that numbers are totally ordered by the ordering, so that two numbers are always comparable, never confused.
- There may by infinitely many left or right options, as long as the recursion is grounded to prevent any infinite play of a game.

Examples of Numbers

- The canonical zero game, with no left or right options at all, is a number that we will call 0.
- The game {0|}, where Left can move to the zero game and Right cannot move, is another number called 1. Its additive inverse {|0} is called -1.
- The Nim game with one pile of one stone can be written {0|0}. It is *not* a number since it has a left option that is $\geq$ one of its right options. Its value is written * or *1.

Examples of Numbers

- What if we add two copies of 1? Left can move to 0 in one copy, leaving a result of 1, and Right cannot move at all. This leads to the game {1|}, which we call 2.
- A Hackenbush stalk with two edges, both blue, gives us the game {1, 0|}, and this is equal to {1|} since the move to 0 is **dominated**.
- Similarly we have games for all integers, both positive and negative, since {n|0} = n+1 and {|-n} = -(n+1), for any positive n.

Examples of Numbers

- Last time we had examples of Hackenbush and Domineering positions where the game amounted to {1|0}. We showed that two copies of such a game have total value 1, so each copy has a value we can call 1/2.
- Similarly {0|1/2} = 1/4 and {1/2|1} = 3/4.
- Note that the value of a game must be greater than all its left options and less than all its right options. In general, the value of a game is the **simplest** number meeting this.

Single-Stalk Hackenbush

- For my own convenience, I'll describe a single-stalk Hackenbush game by a string of B's and R's for the blue and red edges, starting from the ground. (BRBRR for this stalk.)
- Each additional B increases the value, and each R reduces it. Later edges cannot wholly reverse the effect of an earlier one. (For example, if the first edge is B the value remains positive.)
- If we start B^nR the value is between n-1 and n.

Single-Stalk Hackenbush

- Every finite string has a value that is a **dyadic rational**, with denominator a power of two.
- What about infinite stalks? The value would appear to be the limit of the values of the initial segments, so that we can get real numbers like 1/3 with BRRBRBRBRB… as the limit of a sequence of dyadic rationals.

- But while the limit of the sequence 1/2, 1/4, 1/8,… is exactly 0, the value of BRRRRR… is not 0 but positive. Still, it's less than any positive real number! (Left still always wins.)

What are Real Numbers?

- We usually think of real numbers as given by infinite decimal or binary expansions, with the caveat that in binary, expansions like 0.011000… and 0.010111… represent the same real number.
- Formally, irrational reals are defined by **Dedekind cuts** of the rationals. These are partitions of the rationals into two sets L and R where every element of L is less than every element of R.

Single-Stalk Hackenbush

- Every finite single-stalk Hackenbush stalk has a dyadic rational value.
- Every infinite stalk with infinitely many blues and infinitely many reds has a value that is a real number other than a dyadic rational, given by the limit of the initial segment values.
- And every infinite stalk with only finitely many blues represents a value that is infinitesimally larger than a dyadic rational. With finitely many reds it is just a bit smaller.

A Brief History of Calculus

- Newton and Leibniz independently invented calculus in the 1600's. Leibniz' formulation, in particular, used quantities like "dx" and "dy" as if they were real numbers.
- Berkeley in the 1700's pointed out that these infinitesimal objects made no sense, as they disobeyed all the rules for real numbers. Despite having no sound logical basis, though, calculus continued to be used.

A Brief History of Calculus

- In the 1800's, Cauchy, Weierstrass, and Riemann developed the theory of limits (and ε/δ proofs) that we use today.
- In the mid-1900's, Robinson developed a rigorous theory of infinitesimals, essentially a new number system that included both real numbers and infinitesimals. In this system, Leibniz' arguments work perfectly well.

Infinitesimals

- We've seen that the Hackenbush single stalk BRRRRR…. has a positive value that is smaller than that of any positive real number.
- There's also BBBB…. a positive value larger than any real number. And these numbers have additive inverses, and can be added together to get numbers. Two stalks each of BRRR…, for example, have a positive value larger than one BRRR… but still smaller than any positive real number.

Ordinals

- In set theory, the **ordinal numbers** are defined by starting with 0 and applying two rules. You can add 1 to any ordinal, and if you have any sequence $x_0 < x_1 < x_2 < \ldots$ indexed by any ordinal, it has a **limit** larger than all of them.
- The limit of the natural numbers, the first infinite ordinal, is called ω. After that you have $\omega+1$, $\omega+2, \ldots$, which lead to a limit $\omega+\omega = 2\omega$. Then you get 3ω, 4ω, $5\omega, \ldots$, which lead to a limit $\omega \cdot \omega = \omega^2$.

Ordinals

- Similarly you can get ω^2, ω^3, ω^4,…, leading to ω^ω, ω to the power ω^ω, and other things I can't write with this editor.
- All of these ordinals are **countable**. As sets, they have isomorphisms to the natural numbers, but as ordinals they are different because they have no order-preserving isomorphisms.
- The limit of the countable ordinals is called ω_1, the first **uncountable** ordinal.

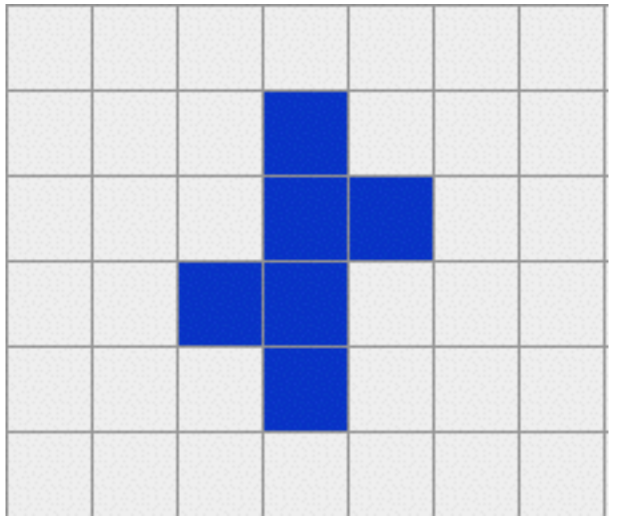
Ordinals as Conway Numbers

- If I have any sequence of numbers $x_1, x_2, x_3, \ldots$, I can define a game where Left can move to any one of them. The resulting game has value that is a number larger than any x_i.
- Thus there is a game (and a number) with any ordinal value, and we can define it as a Hackenbush game with only blue edges.
- But that's only the start of the strangeness.

Some Strange Numbers

- Ordinals can be added and multiplied, but not subtracted or divided. We know that games can be subtracted, and subtracting numbers gives us more numbers.
- That means $\omega-7$ and $\omega^2-39\omega+6$ are perfectly good numbers, values of well-defined games. (We won't do it here, but $\sqrt{\omega}$ can be defined as well, once we have multiplication.)
- Our value for BRRR… turns out to be $1/\omega$.

The Games “Up” and “Down”



- One of the simplest games is {0|*}, where * is the nimber {0|0}. It’s the value of the Domineering position pictured here.
- Because * is not a number, this value ↑, called “up”, is not a number either. But it is positive, and less than every positive number. Its inverse ↓, called “down”, is greater than every negative number but still negative.

Multiplying Numbers

- The numbers have a multiplication operation, making them a field. (Conway says "Field" to emphasize that the numbers are too big to be a set in set theory.)
- Unfortunately this operation, unlike addition, does not have an intuitive meaning in terms of games. But we'll have a go at the definition.

Multiplying Numbers

- Remember that if x^R is a right option of the number x, its value is greater than that of x. And any left option x^L has a smaller value. With numbers, moving always hurts you.
- This means that $(x - x^L)$ and $(x^R - x)$ are positive numbers for any number x.
- If x and y are any two numbers with left options x^L and y^L, and multiplication made any sense, then $(x - x^L)(y - y^L)$ should also be positive.

Multiplying Numbers

- For $(x - x^L)(y - y^L) > 0$ to be true, we must have $xy > x^L y + x y^L - x^L y^L$.
- For $(x - x^L)(y^R - y) > 0$ to be true, we must have $xy < x y^R + x^L y - x^L y^R$.
- Similarly we must have $xy > x^R y + x y^R - x^R y^R$ and $xy < x^R y + x y^L - x^R y^L$.
- We force all four of these to be true by defining $xy = \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid x y^R + x^L y - x^L y^R, x^R y + x y^L - x^R y^L\}$.

Multiplying Numbers

- We force all four of these to be true by defining $xy = \{x^L y + x y^L - x^L y^L, x^R y + x y^R - x^R y^R \mid x y^R + x^L y - x^L y^R, x^R y + x y^L - x^R y^L\}$. Here x^L ranges over all left options, and similarly for the others.
- This defines the multiplication of x any y in terms of other multiplications, each of which involves an option of x or y. Our recursion will eventually reach a base case, and so is well-defined. We can (eventually) prove all the field axioms for this operation with addition.

Multiplying 2 by -1/2

- If $y = 0$, xy is also $0 = \{|\}$ because none of the left or right options exist.
- If $y = 1 = \{0|\}$, $xy = \{x^L y, x^R y\} = \{x^L, x^R\} = x$.
- Let's try multiplying $x = 2 = \{1|\}$ by $y = -1/2 = \{0|-1\}$. All entries with x^R do not exist.
- We have $\{x^L y+xy^L-x^L y^L|xy^R+x^L y-x^L y^R\} = \{-1/2+0-(0)|-2+(-1/2)-(-1)\} = \{-1/2|-3/2\} = -1$.