COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #33: Nim-Type Games
(Tucker Section 10.2)
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Nim-Type Games

- The Game of Nim
- Winning at Nim
- Sums of Games, Zero, Inverses
- Any Nim Game is a Single Pile
- The Sprague-Grundy Theorem
- Proof of the Theorem
- Other Non-Partisan Games

The Game of Nim

- Nim is a two-player non-partisan game of the sort we have been analyzing. A game position is one or more piles, each containing one or more stones. A legal move is to remove one or more stones from a single pile. The winner is the one who removes the last stone.
- The graph of the game starting from position $(x_1,\ldots,x_k)$ has $(x_1+1)(x_2+1)\ldots(x_n+1)$ nodes.

Winning at Nim

- A one-pile game of Nim is easy to analyze. White can win by taking all of the stones.
- With two piles, the winning strategy for White is to equalize the piles if this is possible. If the piles are already equal, Black has a winning strategy by equalizing the piles after each White move.
- Three piles starts to get interesting.

Winning at Nim

- If two (or three) of the three piles are equal, White can win by taking all the third pile.
- From (3, 2, 1), Black has a winning strategy, because every move will leave two equal piles. The same is true of (5, 4, 1) and (6, 5, 3).
- But lots of positions with three different piles are winning for White, such as (4, 2, 1) (move to (3, 2, 1) or (7, 6, 5) (move to (3, 6, 5).

Winning at Nim

- You may have learned the general rule for winning a three-pile game, which is to move to one of these winning positions like (3, 2, 1) if this is possible.
- Winning positions are characterized by the **bitwise XOR** of the three numbers being 0.
- For example, $11 \oplus 10 \oplus 01 = 00$, and $110 \oplus 101 \oplus 011 = 000$ by bitwise XOR.
- To find the right move, XOR the three numbers to get a sum y and then reduce one of the numbers in order to XOR it with y. (We must prove you can.)

Sums of Games

- To explain what is going on here, we'll leave Tucker's presentation and start in on Conway's theory of combinatorial games.
- The most basic notion in that theory is the **sum** of two games. If G and H are any two games of the sort we're studying, G + H is the game where a legal move is to move *either* in G or in H, and a player loses when she has no move in *either* game.

Sums of Games

- The simplest example of a game sum is just Nim itself. The Nim game (4, 3) is the sum of two single-pile Nim games, one with four stones and one with three. The player to move must pick which game to play in, then play in it, and she loses only when she has no moves in any of the games.
- While single-pile games on their own are very boring, when added together they can become interesting.

Zero and Inverse Games

- Why did Conway call this operation "addition"? Because there is a natural notion of a **zero game**: the game with no moves where Black wins immediately, or any other game where Black has a winning strategy. Adding a zero game to any game leaves it the same. (Why?)

- Furthermore, every game G has an **additive inverse** -G, such that G + (-G) = 0.

- In the case of non-partisan G, G's inverse is itself! It is easy for Black to win the game G + G for any G by a **mirror strategy**.

The Mirror Strategy

- If you announced that you have arranged to play two games of chess, one against Magnus Carlsen and the other against Sergei Karjakin,we might think you were crazy.
- But if you can be White against one of them and Black against the other, you have a way to either win one game or draw them both.
- Wait for Carlsen to make his first move as White, then make *that move* against Karjakin.

The Mirror Strategy

- Then when Karjakin replies to you as Black, make *that move* as Black against Carlsen.
- If one grandmaster checkmates you, you will checkmate the other on your next move.
- And if one board becomes a draw, so will the other.
- For a general game G, we let -G be just G with the roles of White and Black reversed. The game G+(-G) is always a win for Black.

Any Nim Game is a Single Pile

- We can show that any game of Nim is "equal" to a one-pile game, where two games G and H are considered equal if $G + (-H) = 0$.
- It suffices to add two arbitrary one-pile games and get a new one-pile game. Equivalently, for any two positive integers x and y, we show that there is a unique non-negative integer such that the Nim game with piles (x, y, z) is a zero game (Black always wins under optimal play.)

Any Nim Game is a Single Pile

- As we've stated before but not yet proved, z is the **bitwise XOR** of x and y. If $x = y$, then, $z = 0$ and we have already shown that $(x, x, 0)$ is a zero game. So assume that $x \neq y$ and that z is the bitwise XOR.
- We must show that Black has a winning response to any White move from (x, y, z). By induction on $x+y+z$, we can assume that any position (x', y', z') with $x'+y'+z' < x+y+z$, and $z = x \oplus y$, is a zero game.

Any Nim Game is a Single Pile

- So assume that White has moved by reducing one of the three numbers, and that the bitwise XOR of the three new numbers is w. All Black needs to do now is alter one of the three new numbers in such a way as to XOR it with w.

- Each of the three numbers has an XOR with w, and we need to show that at least one of these is less than the current number.

- Look at the *highest-order bit* of w, and choose one of the numbers q that has a 1 in this bit. Then $q \oplus w$ must be less than q.

Any Nim Game is a Single Pile

- For example, since $1001 \oplus 1110 = 0111$, the Nim game (9, 14, 7) should be a zero game.

- Suppose White reduces the 14 to an 8. Here w $= 1110 \oplus 1000 = 0110$, or 6.

- Black's winning response in (9, 8, 7) is to XOR one of the three numbers with 6. He can't change 9 to $1001 \oplus 0110 = 1111$, or change 8 to $1000 \oplus 0110 = 1110$, but he can change 7 to $0111 \oplus 0110 = 1$ and (9, 8, 1) is a zero game.

- At least one number had to have a 1 in the 2^2 place, since the three XOR'd to a 1 there.

The Sprague-Grundy Theorem

- The **Sprague-Grundy Theorem** says that any finite impartial game (of the type we are discussing) is equivalent to a one-pile Nim game.
- We have just seen that the one-pile games are closed under addition, but there are other ways to define impartial games.
- Tucker's graph formulation is general. Another formulation is to say that a game is a start position with a finite set of **successor** games, where White can choose any game in the set.

Proof of the Theorem

- We can use induction on this latter recursive definition (or just induction on the number of nodes in the directed graph) to prove the Sprague-Grundy Theorem.
- The base case is an empty game, which is clearly equivalent to a size-0 Nim pile.
- If White can choose to move to G_1, G_2,…, or G_k, and each G_i is equivalent to a Nim pile, we will show the new game is equivalent to a pile whose size is the Grundy function of the G_i's.

Proof of the Theorem

- Let the Nim-equivalent of each game G_i be n_i, so that g is the least number not equal to any n_i.
- To show that a Nim-pile of g is equivalent to the given game G, we show that $G + g = 0$, where g here represents that Nim game.
- White's first move is either to $G + x$ for some $x < g$, or to $n_i + g$ for some i. We must show that each of these moves loses.
- Remember that these + signs are *game addition*.

Proof of the Theorem

- From G + x Black can move to x + x, a winning position. This is because every number less than g is equal to n_i for some i, and so is an option in the game G.
- If White moves to $n_i + g$, we have two possible cases. If $n_i > g$, Black moves to g + g and wins. If $n_i < g$ Black moves to $n_i + n_i$ and also wins.
- This completes the proof of the Sprague-Grundy Theorem, at least for finite impartial games.

Kayles

- **Kayles** is a bowling game where a player can knock down a single pin or a pair of adjacent pins. The last player to bowl wins.

- We assume they are skilled enough to knock down exactly what they want to. What is the best strategy if we start with a row of n pins?

THE GAME OF KAYLES

This was introduced by Dudeney and Loyd. It is played by skilful players with a number of rows of ninepins. See Fig. 26.

Fig. 26. The Kayles position $K_4 + K_1 + K_2 + K_5$.

We suppose the players are so skilful that they can throw a ball so as to knock down any single ninepin or any two adjacent ones, but that it is physically impossible to knock down pins separated by any larger distance. The last mover wins.

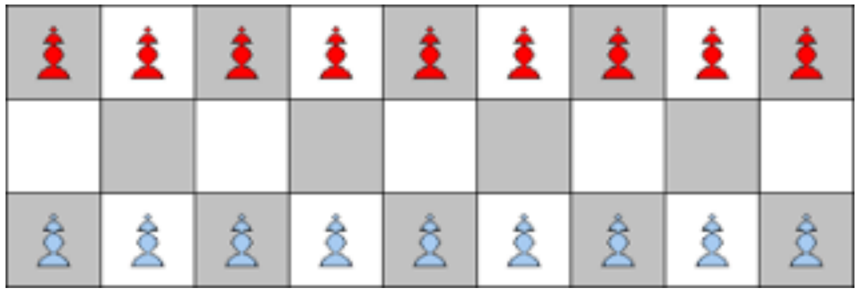
Kayles

- Clearly if n = 0 we have a zero game, and if n = 1 or n = 2 White can win by knocking down all the pins. The n = 1 game is like Nim with one stone, and n = 2 is like Nim with one pile of two stones.
- With n = 3, White has essentially three moves, to 2 by taking an outside pin, to 1 + 1 by taking the middle pin alone, or to 1 by taking two pins. Just as in Nim, 1 + 1 = 0, so 3 must be losing. In fact 3 has a Nim-value of 3, the least Nim-number you can't move to from it.

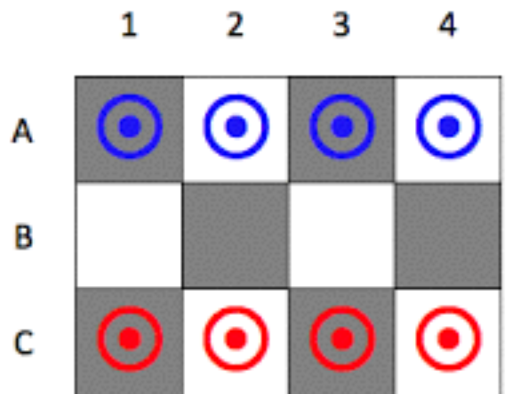
Kayles

- With n = 4, White can move to 3, 2, 1 + 2, or 1 + 1. So 4 is another losing position. The Nim values of the possible outcomes are 3, 2, 3, and 0, so the Nim-value with n = 4 is 1.
- With n = 5, White can move to 4, 3, 1 + 3, 1 + 2, 2 + 1, or 2 + 2. The Nim-values of these are 1, 3, 2, 3, and 0 respectively, so the Nim-value with n = 5 is 4.
- For n > 70, the value of n is periodic with period 12 (see Wikipedia "Kayles").

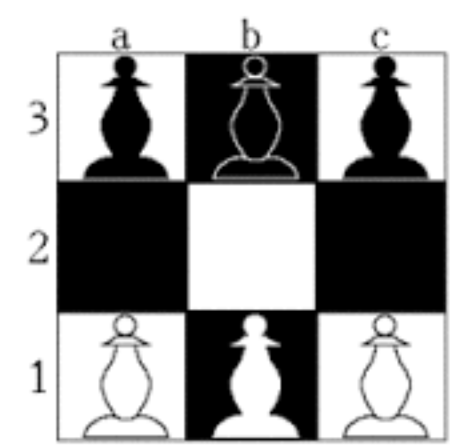
Dawson's Chess



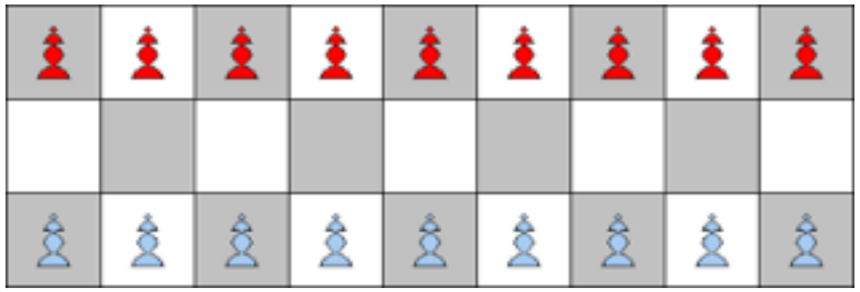
- **Dawson's Chess** is played on a 3 by n board, where each side has n pawns. Pawns move as in chess, forward one or diagonally to capture.



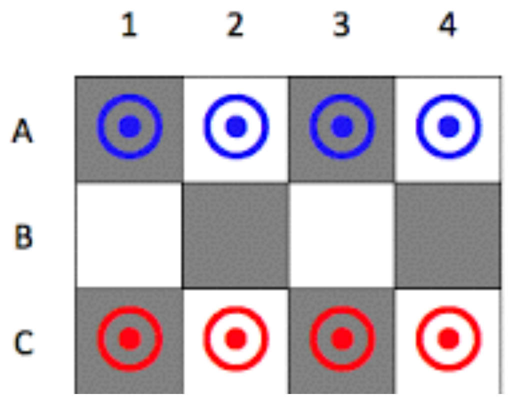
- Captures are **mandatory**, which makes the game less interesting to play but fun to analyze.



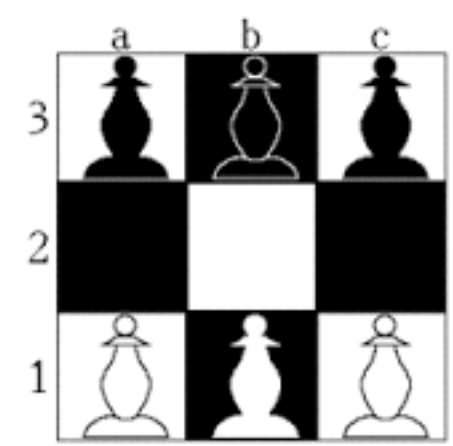
Dawson's Chess



- Any pawn push initiates a bloodbath, which takes out the pawn pushed and pawns on either side of it, leaving two isolated pawns that cannot ever move again.



- This makes Dawson's Chess surprisingly similar to Kayles.



Dawson's Kayles

- Consider the variant of Kayles where the player can take out (a) a single isolated pin, (b) two adjacent pins at the end of a row, or (3) three adjacent pins anywhere. Let D_n be this game with a row of n, a game equivalent to Dawson's Chess with n pawns each.
- D_0 is a zero game, and D_1 and D_2 each have a Nim-value of 1. In D_3, the options are 0 (by taking all three) or 1 (by taking 2) so the Nim-value of D_3 is 2.

Dawson's Kayles

- In D_4 White can leave D_2 or D_1, each of which has Nim-value 1, so the value of D_4 is 0.
- In D_5, White can leave D_3, D_2, or D_1+D_1, which have values 2, 1, and 0 respectively, so the value of D_5 is 3.
- In D_6, White can leave D_4, D_3, or D_1+D_2, which have values 0, 2, and 0, so D_6 has value 1.
- These values are also eventually periodic!