COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #31: Polya's Formula
(Tucker Section 9.4)
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Polya’s Formula

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Pattern Inventories

- We'll finish our study of groups and symmetries by looking at **pattern inventories** for colorings.
- We've learned how to count equivalence classes of colorings defined by a group of symmetries.
- But the set of all colorings is already divided into subsets, based on the number of occurrences of each color.

Pattern Inventories

- With 2-colorings of the four vertices of a square, we could have 0, 1, 2, 3, or 4 white vertices, and no symmetry can alter that number. So there are at least five equivalence classes, and we saw already that there are 6.
- A more complete description than "six classes" is "one class with 0 whites, one with 1, two with 2, one with 3, and one with 4".
- We can write this is as the polynomial $b^4 + b^3w + 2b^2w^2 + bw^3 + w^4$, the pattern inventory.

Two-Colored Squares Again

- Let's see how we can calculate this pattern inventory in the case of 2-colored squares.
- The identity fixes *all* the colorings, and we can inventory *all* the colorings by the polynomial $b^4 + 4b^3w + 6b^2w^2 + 4bw^3 + w^4$. Note that this equals $(b+w)^4$.
- The two rotations fix only the mono-colored colorings, which have inventory $b^4 + w^4$.

Two-Colored Squares Again

- The three double-flips each fix four colorings, inventoried by $b^4 + 2b^2w^2 + w^4 = (b^2+w^2)^2$.
- The two single flips each fix eight colorings, inventoried by $b^4 + 2b^3w + 2b^2w^2 + 2bw^3 + w^4 = (b+w)^2(b^2+w^2)$.
- Adding the eight inventories (one for each permutation in the group) gives us the inventory $8b^4 + 8b^3w + 16b^2w^2 + 8bw^3 + 8w^4$. Dividing this by 8 gives us the overall pattern inventory.

What's Going On?

- We know that when we add up the elements fixed by each permutation, we get |G| copies from each equivalence class. This is why the cycle index polynomial, evaluated with r for each variable, gives us the number of classes.
- What's happening now is that we are replacing each of those r's by b's and w's, so that each monomial in the eventual sum becomes a monomial in b's and w's, marking the number of uses of each color in that coloring.

What's Going On?

- The identity permutation, for example, has four 1-cycles and fixes all 2^4 colorings. The polynomial $(b+w)^4$ has one monomial for each of these colorings.
- A double-flip, by contrast, has two 2-cycles, and a color is assigned to each 2-cycle in a coloring fixed by it. Since each cycle has two blacks or two whites, the polynomial $(b^2+w^2)^2$ inventories those fixed colorings.

Polya's Formula

- Recall that the cycle index polynomial P_G for a group G is 1/|G| times the sum, for each element π of G, of a monomial giving the cycle structure of π.
- Polya's theorem says that if we substitute b +w for x_1, b^2+w^2 for x_2, and similarly b^k+w^k for each x_k, and evaluate P_G with those values, we get the pattern inventory for 2-colorings.
- With more than two colors we use the sum of the k^{th} powers of a variable for each color.

Rotations of a Triangle

- Let's look at this with G as $\mathbb{Z}_3$ and S as a triangle, so that G is the group of rotations.
- The cycle index polynomial is $(x_1^3+2x_3)/3$, and substituting we get a pattern inventory of $((b+w)^3+2(b^3+w^3))/3 = b^3+b^2w+bw^2+w^3$. This represents the four classes of 2-colorings.
- With three colors we get $((b+w+r)^3+2(b^3+w^3+r^3))/3 = b^3+w^3+r^3+b^2w+b^2r+w^2b+w^2r+r^2b+r^2w+2bwr$. The number of colors determines the class except for bwr.

Rotations of a Heptagon

- With a 7-gon the cycle index polynomial for rotations is $(x_1{}^7+6x_7)/7$, so Polya's formula for 2-colorings gives us $((b+w)^7+6(b^7+w^7))/7$.
- The series of coefficients for $(b+w)^7$ is a line of Pascal's Triangle, (1 7 21 35 35 21 7 1). The other term has coefficients (6 0 0 0 0 0 0 6), so the sum is (7 7 21 35 35 21 7 7). Dividing by 7 and reverting to polynomial notation gives $b^7+b^6w+3b^5w^2+5b^4w^3+5b^3w^4+3b^2w^5+bw^6+w^7$. There are 20 total classes and we have the pattern inventory.

Edges of a Tetrahedron

- The group A_4 of symmetries of a tetrahedron also acts on the six *edges* of the tetrahedron. The cycle index polynomial for that action is $(x_1^6+8x_3^2+3x_1^2x_2^2)/12$, as we can see by analyzing the eight 120-degree rotations about a point and the three double-flips.
- Substituting (b^k+w^k) for x_k gives us the sum of three polynomials with coefficients (1 6 15 20 15 6 1), (8 0 0 16 0 0 8), and (3 6 9 12 9 6 3). The sum is (12 12 24 48 24 12 12) and the pattern inventory is $b^6+b^5w+2b^4w^2+4b^3w^3+2b^2w^4+bw^5+w^6$.

Vertices of a Cube

- One more example is the symmetries of a cube. There are 24, because we could have any of the six sides on the bottom in any of four orientations. The group is isomorphic to S_4, but is acting on the eight vertices.

- The cycle index polynomial takes some work to compute: $(x_1^8+6x_4^2+9x_2^4+8x_1^2x_3^2)/24$.

- We add (1 8 28 56 70 56 28 8 1), (6 0 0 0 12 0 0 0 6), (9 0 36 0 54 0 36 0 9), and (8 16 8 16 16 32 16 8 16 8), divide by 24, and get $b^8+b^7w +3b^2w^6+3b^3w^5+7b^4w^4+3b^3w^5+3b^2w^6+bw^7+w^8$.

Undirected Graphs

- Consider the set of possible edges in an n-vertex undirected graph. A permutation of the vertices also permutes the edges. So we can think of the group S_n as acting on the edges, with a cycle index polynomial.
- A particular undirected graph can be thought of as a two-coloring of the edges, with one color for "edge" and one for "non-edge". (This is the basis of Exercise 9.4.16 on HW#7.)

Undirected Graphs

- Two n-vertex graphs are isomorphic if there is a permutation of the vertices that takes one to the other. Thus the number of graphs, up to isomorphism, is the number of 2-colorings of the edge set, up to the action of S_n on that edge set, and can be computed by the methods we've used here.
- Polya's Formula can give us a pattern inventory of these "colorings", which is an inventory of the graphs by number of edges.

Undirected Graphs

- In 2007 I wanted a list of all the graphs of various small sizes, up to isomorphism. I also wanted a list of "two-colored graphs", which correspond to 3-colorings of the edge set with "no edge", "red edge", and "blue edge".
- I wrote a computer program to generate these lists, and the results are on my web site. I just made a backtrack search through the possibilities, rejecting any graph that was isomorphic to a lexicographically smaller graph.

Undirected Graphs

- Once you have solved Exercise 9.4.16, you will know how to use Polya's Formula to get pattern inventories for each of these lists of graphs.
- Of course to construct the inventories you would need the cycle index polynomial of the action of S_n on the edges, for each n.
- For S_3 the action on the three edges is the same as that on the three vertices. But for S_4 things already get more interesting.