COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #30: The Cycle Index
(Tucker Section 9.3)
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The Cycle Index

- Review Burnside's Theorem
- Colorings of Squares Again
- Cycles and the Number Fixed
- The Cycle Index Polynomial
- The Cycle Index Theorem
- Colorings of an n-Gon
- Colorings of a Tetrahedron

Burnside's Theorem

- We've just proved two versions of a theorem relating the number N of equivalence classes of colorings to the number of colorings fixed by elements of a symmetry group G.
- N is $1/|G|$ times the number of pairs (x, π) where π is a permutation in G that fixes x.
- We can count this number as the sum of $\varphi(x)$ for all x or as the sum of $\Psi(\pi)$ for all π.

Colorings of Squares Again

- Let's try to apply this theorem to count the number of r-colorings of a square under the action of the dihedral group of rotations and reflections.

- We have eight permutations. What do they do with r=2? The identity fixes all 16. The two 90 degree rotations fix only two. Three fix four, and two fix eight!

- Note that Tucker's Figure 9.7 has typos.

Cycles and the Number Fixed

- Call the corners of the square a, b, c, and d in cyclic order. In cycle notation, we can write the eight elements of G as 1, (abcd), (ac)(bd), (adcb), (ab)(cd), (ad)(bc), (ac), and (bd).
- To be fixed by a particular permutation, a coloring must have the same color for every vertex in a cycle of that permutation.
- Thus 1 (the product of four 1-cycles) fixes any coloring, but (abcd) fixes only r of them.

The Cycle Index Polynomial

- What this means is that the fixed-point behavior of a permutation depends only on its **cycle structure**, which is the number of cycles of each size.
- We can represent the cycle structure of a group as a polynomial, with variables $x_1,\ldots,x_{|S|}$ and each variable x_i appearing to a power equal to the number of i-cycles in a particular permutation. We include cycles of length 1.

Cycle Index Examples

- We have a monomial for each element of the group, and we divide the sum of these by |G|.
- The 1-element group $\mathbb{Z}_1$ has cycle index x_1.
- The 2-element group $\mathbb{Z}_2$ has cycle index $(x_1^2+x_2)/2$, as one permutation has two 1-cycles and the other one 2-cycle.
- The 3-element group $\mathbb{Z}_3$ has cycle index $(x_1^3+2x_3)/3$, for the identity with three 1-cycles and the two others each with one 3-cycle.

Cycle Index Examples

- There are two groups with four elements, $\mathbb{Z}_4$ with cycle index $(x_1^4+x_2^2+2x_4)/4$, and $\mathbb{Z}_2\times\mathbb{Z}_2$ with cycle index $(x_1^4+3x_2^2)/4$.
- The only group with five elements is $\mathbb{Z}_5$, with cycle index $(x_1^5+4x_5)/5$. In general for prime p, $\mathbb{Z}_p$ is the only group with p elements and has cycle index $(x_1^p+(p-1)x_p)/p$.
- The two groups with six elements are $\mathbb{Z}_6$ with index $(x_1^6+x_2^3+2x_3^2+2x_6)/6$ and S_3 with cycle index $(x_1^3+3x_1x_2+2x_3)/6$.

Groups of Permutations

- In algebra we consider two groups to be the same if they are **isomorphic**, meaning that there is a **group homomorphism** from one to the other that is a bijection. A group homomorphism is a map f such that f(xy) always equals f(x)f(y).
- But the cycle index is not preserved by group isomorphism, as it depends on how the group acts as a group of permutations of some finite set.

Groups of Permutations

- S_3 is the group of all permutations of a three-element set: {1, (ab), (ac), (bc), (abc), (acb)}.
- But any group can be represented as a group of permutations of *itself*, by having y take each x to xy. If we call the elements of S_3 {a,b,c,d,e,f}, the six permutations can be written 1, (ab)(cf)(de), (ac)(be)(df), (ad)(bf)(ce), (aef)(bcd), and (afe)(bdc).
- Here the cycle index is $(x_1^6+3x_2^3+2x_3^2)/6$.

The Cycle Index Theorem

- We observed earlier that a permutation with k disjoint cycles fixes any coloring that has a common color for each cycle, so it fixes exactly r^k colorings.
- If we substitute the value r for each of the variables $x_1,\ldots,x_n$, each monomial representing a permutation with k cycles contributes r^k to the sum. Thus this value of the cycle index polynomial is exactly $(1/|G|)$ times the sum over all π of $\Psi(\pi)$, which by Burnside's Theorem is exactly N.

The Cycle Index Theorem

- Thus for any set S, and for any group G of permutations of S with cycle index polynomial $P_G(x_1,\ldots,x_n)$, we have that the number of nonequivalent m-colorings of S is given by $P_G(m,\ldots,m)$.
- For the dihedral group on the square, we had $P_G(x_1,x_2,x_3,x_4) = (x_1^4+2x_1^2x_2+3x_2^2+2x_4)/8$. This gives us $P_G(2,2,2,2) = (16+16+12+4)/8=6$, $P_G(3,3,3,3) = (81+54+27+6)/8 = 21$, and $P_G(4,4,4,4) = (256+128+48+8)/8 = 55$.

Batons Revisited

- Recall our example of k-banded batons, with a two-element G consisting of the identity and a flip. The cycle index polynomial for even k is $(x_1^k+x_2^{k/2})/2$, and for odd k is $(x_1^k+x_1x_2^{(k-1)/2})/2$.
- To get the number of r-colorings, we simply substitute r for x_1 and x_2 to get $(r^k+r^{k/2})/2$ in the case of even k and $(r^k+r^{(k+1)/2})/2$ in the case of odd k.

Colorings of an n-Gon

- A one-sided n-gon has $\mathbb{Z}_n$ as its group of symmetries, as reflections are not permitted.
- For prime n, $P_G(x_1,\ldots,x_n) = (x_1{}^n+(n-1)x_n)/n$, and thus the number of r-colorings is $(r^n+(n-1)r)/n$.
- For composite n things are more complicated. For n=8, for example, $P_G(x_1,\ldots,x_n) = (x_1{}^8+x_2{}^4+2x_4{}^2+4x_8)/8$, and thus $N = P_G(r,\ldots,r) = (r^8+r^4+2r^2+4r)/8$, which when r=2 is $(256+16+8+8)/8 = 36$.

Coloring a Tetrahedron

- We observed earlier that a regular tetrahedron has twelve symmetries, as any of the four faces may be on the bottom in any of three orientations.
- The group {1, (abc), (acb), (abd), (adb), (acd), (adc), (bcd), (bdc), (ab)(cd), (ac)(bd), (ad)(bc)} is called A_4 because it consists of all the **even** permutations of {a,b,c,d}. (Even means the product of an even number of **transpositions**: we would need to prove this well-defined.)

Coloring a Tetrahedron

- By inspection, the cycle index of A_4 is $(x_1^4+8x_1x_3+3x_2^2)/12$.
- This means that the number of 2-colorings of a tetrahedron up to symmetry is $(2^4+8(2^2)+3(2^2))/12 = (16+32+12)/12 = 5$. This works because any two colorings with the same number of white nodes are the same.
- For 3-colorings we have $(81+8(9)+3(9))/12 = 15$. Again the number of nodes of each color suffices to determine the equivalence class.

The Group S_n

- We defined S_n to be the group of *all* permutations of n objects, with n! elements.
- Under S_n, two r-colorings are equivalent if and only if they have the same number of objects of each color, so we know there are $C(n+r-1, r-1) = C(n+r-1, n)$ equivalence classes.
- Evaluating the cycle index $(x_1^3+3x_1x_2+2x_3)/6$ at (r,r,r) gives us $(r^3+3r^2+2r)/6$ which is exactly $C(3+r-1, 3)$.

The Groups S_4 and S_5

- To get the cycle index of S_4, we need to classify all the permutations by cycle structure: $(x_1^4+6x_1^2x_2+3x_2x_2+8x_1x_3+6x_4)/24$.
- The r-colorings of a set thus number $(r^4+6r^3+11r^2+6r)/24$, and this number is just $C(r+3, 4)$.
- The possible cycle structures in S_5 may be familiar as poker hands. The cycle index is $(x_1^5+10x_1^4x_2+15x_1x_2^2+20x_1^2x_3+20x_2x_3+30x_1x_4$ $+24x_5)/120$.