COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #3: Isomorphism and Edge Counting
(Tucker 1.2, 1.3)
David Mix Barrington
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Isomorphism and Edge Counting

- Definition of Isomorphism
- Some Special Graphs
- Testing for Isomorphism
- Counting Edges and Degrees
- The Mountain Climbers Puzzle
- Bipartite Graphs and Odd Circuits
- Testing for Bipartiteness

Definition of Isomorphism

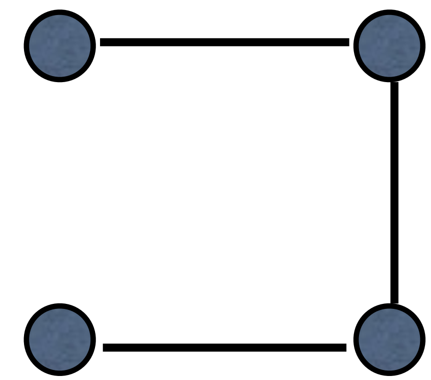
- Two graphs are **identical** if they have the same vertex set and the same edge set.
- Two different graphs are called **isomorphic** if they are “essentially the same”.
- Formally, G and H are isomorphic if there is a function f from the vertices of G to the vertices of H that is a bijection, so that (x, y) is an edge of G if and only if (f(x), f(y)) is an edge of H.

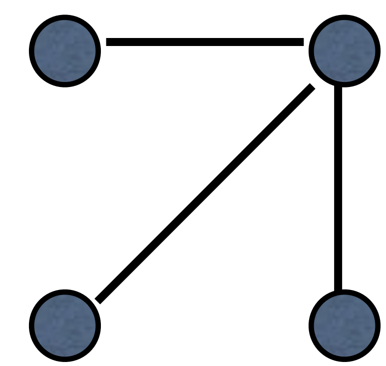
Isomorphism of Small Graphs

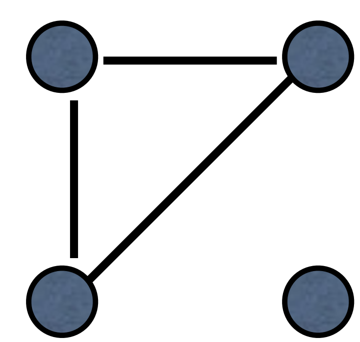
- To be isomorphic, two graphs must have the same number of vertices and the same number of edges.
- All one-vertex graphs are isomorphic.
- There are two classes of two-vertex graphs, one with an edge and one without.
- Three-vertex graphs are also isomorphic if they have the same number of edges.

Four-Vertex Graphs

- Four-vertex graphs may have from 0 to 6 edges.
- The graphs with 0, 1, 5, and 6 form one class each.
- There are two classes with 2 or 4, and three with 3.







Some Special Graphs

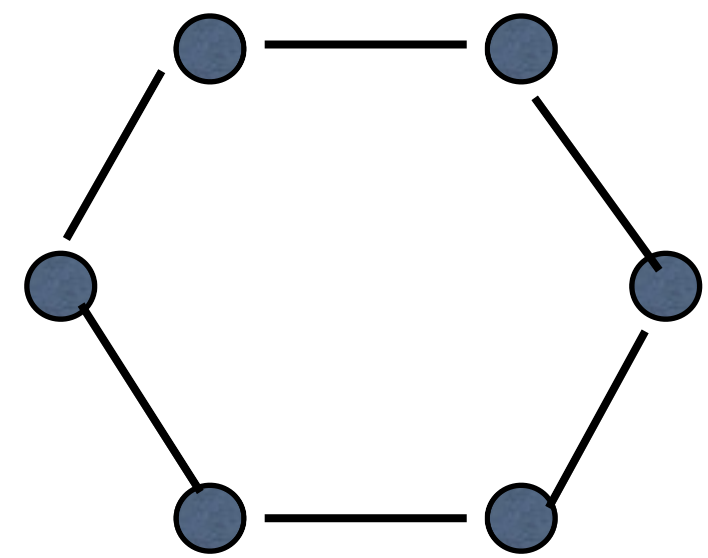
- Given any number n of vertices, there is a **complete graph** K_n with all $(n^2 - n)/2$ possible edges, and K_n's **complement** with no edges.
- The **cycle graph** C_n be thought of as having vertex set {0,…,n-1} and edges (i, (i+1)%n) for each i.
- The **star graph** S_n has the same vertex set and edges (0, i) for all i in {1,…,n}.

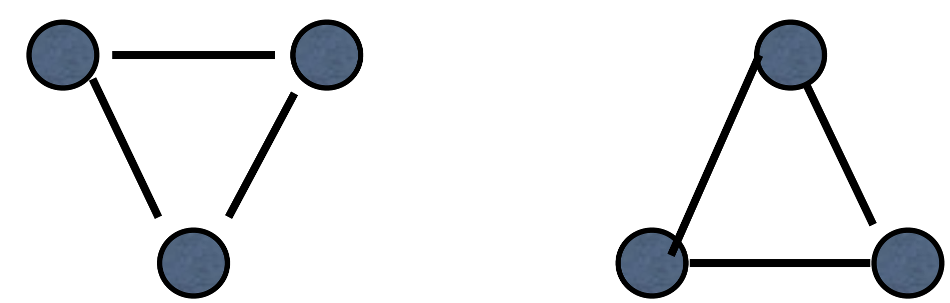
Testing for Isomorphism

- The first test for isomorphism is the number of nodes and edges.
- If those match, we can next look at the number of nodes of each **degree**. For example, our three graphs with four nodes and three edges had degree sequences (0, 2, 2, 2), (1, 1, 1, 3), and (1, 1, 2, 2) and thus no two of them can be isomorphic.

Degrees are Not Enough

- Here's an example of two graphs, each with six nodes and six edges. Both have the degree sequence (2,2,2,2,2,2), but the graphs are not isomorphic.
- Of course, the top graph has no triangle, where the bottom graph has.





Counting Edges and Degrees

- If we add the degrees of all the vertices, we get twice the number of edges since each edge contributes 1 to each endpoint.
- Hence the number of vertices of odd degree in a graph, or even in a **connected component**, must be even.
- Some degree sequences are impossible. We can't have a set of seven people, each of whom knows exactly three of the others.

Testing for Isomorphism

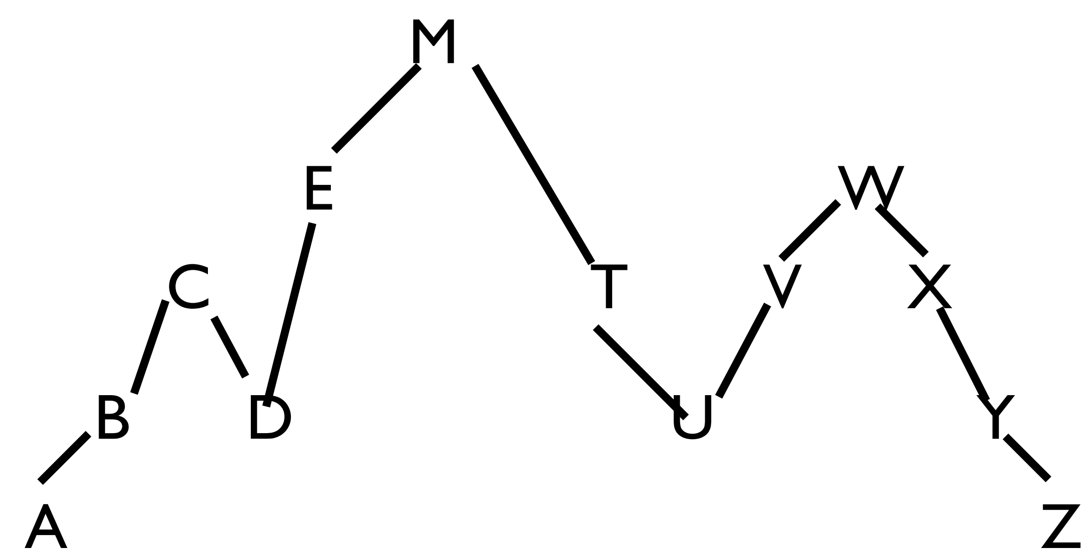
- Tucker talks about the "AC principle", where you make Assumptions and work out their Consequences. If the consequences include a contradiction, the assumptions are false.
- If an isomorphism exists, it must take each node x to a node f(x) of the same degree. If there is only one node of the right degree, you know that it is f(x). Then x's neighbors must map to neighbors of f(x).

Testing for Isomorphism

- If you have no choice, you proceed in constructing your candidate isomorphism. If you have choices, you can break into cases.
- Eventually you will either construct an isomorphism or show that none exists in any of your cases.
- It is not known whether there is a poly-time algorithm to test isomorphism in general, but the problem is believed *not* to be NP-complete.

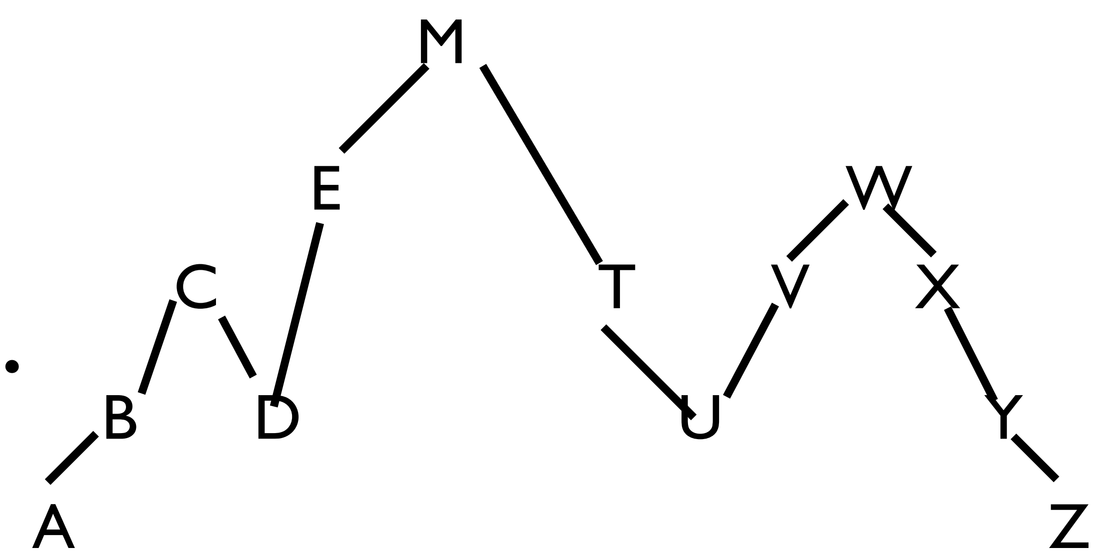
The Mountain Climbers Puzzle

- Two climbers, one starting at A and one at Z, want to meet at M while always being at the same elevation as one another.
- We can show that this is always possible.

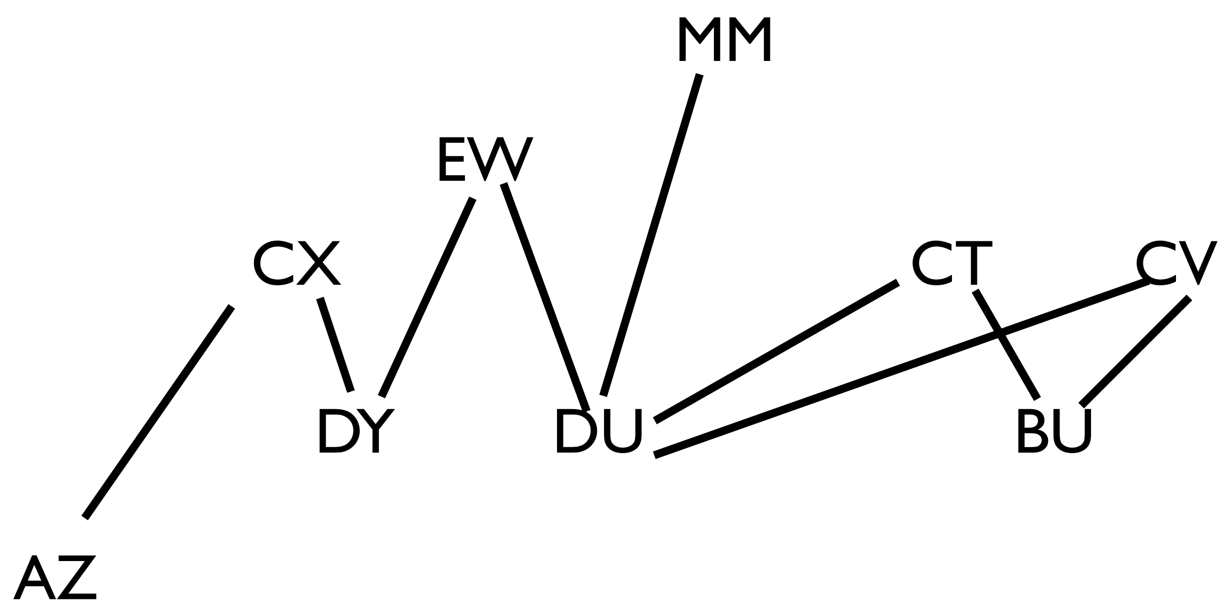


The Mountain Climbers Puzzle

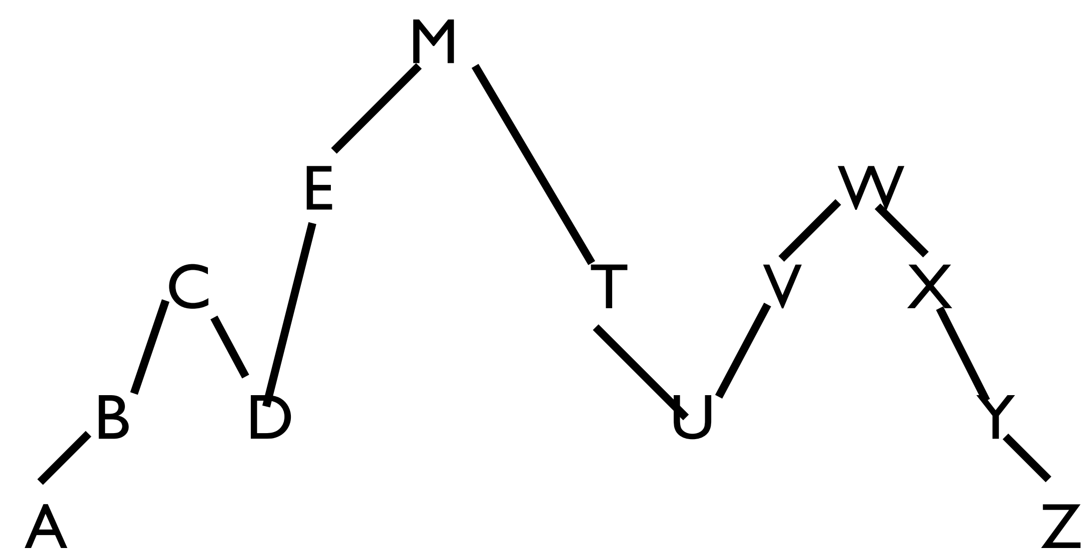
- A state of the system is a pair of nodes, one for each climber, where at least one is a peak or valley.
- We need to show that they can get from (A, Z) to (M, M).
- Two pairs are connected if they can get from one to the other while matching elevation.



The Mountain Climbers Puzzle



- A win is a path from AZ to MM in this new graph.
- But every node except AZ and MM has even degree.
- A component must have a degree sum that is odd.



Bipartite Graphs & Odd Cycles

- Remember our example where the nodes were divided into two sets X and Y, and each edge has endpoints in both X and Y.
- A graph is **bipartite** if is is possible to do this.
- A graph is bipartite if and only if every circuit in it has even length.
- For one half, note that if we are bipartite, any circuit starting at X must go X-Y-X-Y-…-X, and be even.

Testing for Bipartiteness

- For the other half, given an arbitrary graph, we can greedily mark its nodes as being in X or Y. Pick any node to be in X, put its neighbors in Y, the neighbors of those in X, and so on. If you finish a component and there are nodes left, pick a new node to be in X.
- If you succeed, the graph is bipartite. If you fail, you have found an odd-length circuit. So if there are no such circuits, you will succeed.