COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #28: Equivalence and Symmetry Groups
(Tucker Section 9.1)
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Equivalence, Symmetry Groups

- Colorings of a Square
- Equivalence Relations on Colorings
- Symmetries of Regular Polygons
- Permutations and Cycle Notation
- Groups of Permutations
- Partitions Induced By a Group

Symmetries and Games

- In this last part of the course, we'll be looking at two new kinds of combinatorial problems.
- Chapter 9 of Tucker deals with **symmetries**. How do we count objects when the same object may have multiple presentations?
- Chapter 10, and some additional material I'll provide, deals with the **combinatorial game theory** developed by Conway.

Colorings of a Square

- Suppose I color the vertices of a square. If I assign one of r colors to each vertex, there are r^4 ways to do this.
- But if I rotate the square, or flip it over, I can turn one of these colorings into another. If I consider two colorings to be “the same” if this is possible, then my new number of colorings is less than r^4 and more difficult (and interesting) to calculate.

Colorings of a Square

- Let's look at the 16 2-colorings of a square.
- I'll denote a coloring by giving the colors of the vertices, clockwise from the top left. So they range from WWWW to BBBB.
- As I rotate or reflect, one thing that I cannot change is the number of white and black vertices. There is one coloring with four whites, four with three, six with two, four with one, and one with none.

Colorings of a Square

- The four colorings with three whites may be mapped to one another by a rotation, as can the four colorings with one white.
- But no rotation can change WWBB to WBWB, so there are two kinds of colorings with two whites. Every coloring with two whites can be rotated to one of these.
- The number of colorings "up to symmetry" is thus 1+1+2+1+1 = 6.

Equivalence Relations

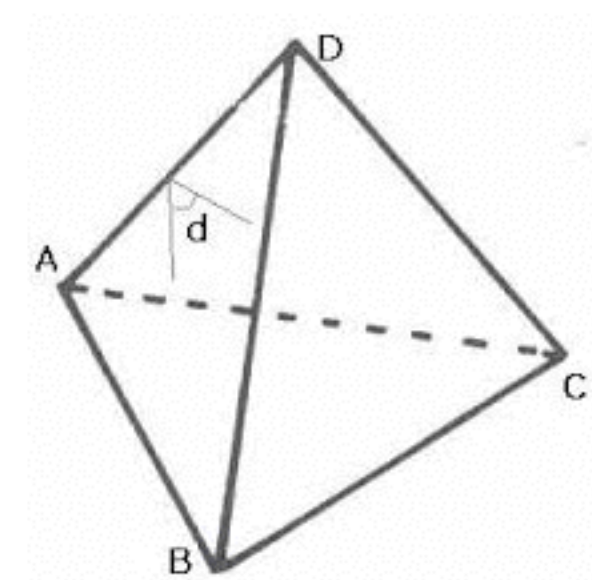
- We know from COMPSCI 250 that a binary relation that is **reflexive**, **symmetric**, and **transitive**, called an **equivalence relation**, divides its base set into **equivalence classes**.
- If x is any element of A, its equivalence class [x] is the set $\{y: (x, y) \in R\}$. The equivalence classes form a **partition** of A, and the "same-set" relation of any partition is an equivalence relation.

Symmetries of Regular Polygons

- A **symmetry** of a geometrical figure is a map from the figure to itself.
- In the case of a polygon, we can view a symmetry as a function from the vertices to the vertices, where f(v) is the vertex to which we move vertex v.
- A regular polygon with n sides (an n-gon) has 2n possible symmetries. We can rotate it by any multiple of 360/n degrees, or flip it over.

Symmetries of a Tetrahedron

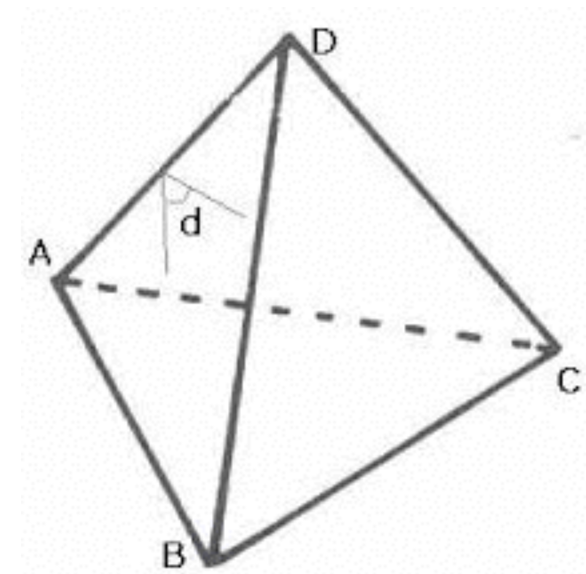
- A regular tetrahedron can be placed in twelve possible ways. We can decide which of the four faces will go on the bottom, and then which of the three orientations that bottom face (a regular triangle) will be in.



- Any of these orientations may be moved to any other by a rotation in three dimensions.

Symmetries of a Tetrahedron

- Any of these symmetries can be thought of as a function (a bijection) from the set of vertices to itself.
- Rotating the bottom clockwise (looking down) would take A to C, C to B, B to A, and leave D fixed.
- Rotating in a different plane could take A to C, B to D, C to A, and D to B.



Permutations

- Each of the twelve symmetries corresponds to a bijection from the set {A, B, C, D} to itself, called a **permutation** of the set.
- If we represent a bijection f by the string f(A)f(B)f(C)f(D), then these twelve turn out to be ABCD, ACDB, ADBC, BADC, BCAD, BDCA, CADB, CBAD, CDBA, DACB, DBAC, and DCBA.
- This is half of the 4! = 24 possible bijections.

Cycle Notation for Permutations

- This is called the **pointwise** notation for permutations. There is another notation.
- Consider permutation BDAC. If I apply this permutation to A I get B. If I apply it again to B I get D, and applying it more gets me C, and then A again. We call this sequence A-B-D-C-A a **cycle**. Any permutation takes any element through a cycle, and these cycles form a partition of the base set.

Cycle Notation for Permutations

- The **cycle notation** for a permutation lists the cycles, with the elements of each cycle in their cyclic order. So this permutation that was called BDAC in pointwise notation is now called (ABDC) in cycle notation. The point wise permutation CDAB is called (AC)(BD).
- By convention, we start each cycle with its smallest member, and order the cycles in descending order of smallest member.

Composing Permutations

- If f and g are two permutations of the same set X, there is a new permutation h that we obtain by first performing f, then performing g. (Of course the composition of two bijections is a bijection.) We call this the **product** of f and g and write it "fg".
- It is fairly easy to multiply permutations when they are given in cycle notation.

Composing Permutations

- Let f be (AC)(BD) and g be (ABDC). The product of f and g can be written (AC)(BD)(ABDC), as a product of permutations that are each a single cycle. (We generally leave out cycles of size 1, so (C)(ADB) would be written just (ADB).
- We'd like to write (AC)(BD)(ABDC) as a product of disjoint cycles, a normal form for cycle notation. How do we do this?

Composing Permutations

- Looking at (AC)(BD)(ABDC), where does A go? The first cycle takes it to C, the second leaves it as C, and the third takes it back to A. So the product fixes A.
- B is fixed by the first, goes to D by the second, and to C by the third. C goes to A by the first, is fixed by the second, and goes to B by the third. So the product has the cycle (BC). And we can confirm that the product also fixes D, so the product *is* just (BC).

Groups of Permutations

- A **group** is a set of objects G with a binary operation $(G \times G) \rightarrow G$, usually called a **product**, that is **associative**, has an **identity element**, and has an **inverse** for every element.
- Any nonempty set of permutations of a finite set that is **closed** under composition must form a group, with composition as the operation.
- Let's prove this claim.

Groups of Permutations

- Let G be any nonempty closed set of permutations of a finite set and let f be any permutation in G.
- If we look at f, f^2, f^3,…, where f^k is the composition of k copies of f, we must eventually have $f^i = f^j$ for some i and j with $i \neq j$, since the set is finite and has only finitely many permutations.
- But then f^{j-i} is the identity permutation and is in G. And $f(f^{j-i-1}) = f^{j-i}$ = the identity, so f^{j-i-1} is the inverse of f and is in G.

Partitions Induced by a Group

- Now consider colorings of a set of vertices, and some group G of permutations of the vertices. (For example, G might be the set of 12 symmetries of the tetrahedron, viewed as permutations of the vertices.)
- The group has an **action** on the colorings, because permuting the vertices leads to a new coloring. (Or, in some cases, to the same coloring.)

Partitions Induced by a Group

- Given a group G, we can define an equivalence relation on the colorings, where $R(c_1, c_2)$ is true if there exists some f in G that takes c_1 to c_2.
- If G is the group of symmetries we defined earlier, this equivalence relation is the property of being "the same coloring" we discussed at the start of this lecture.
- It partitions the colorings into classes.

Partitions Induced by a Group

- So under the group of eight symmetries of the square, the 16 2-colorings of the square are partitioned into six equivalence classes.
- What about the 81 3-colorings of the square, under the same group? All the elements of a class must have the same multiset of vertex colors, but not all colorings with the same multiset will be equivalent. We'll look at how to count the classes after the break.