COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #27: Rook Polynomials
(Tucker Section 8.3)
David Mix Barrington
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Rook Polynomials

- Finding Matchings With Excluded Positions
- An Example With Five Positions
- Placing Non-Capturing Rooks
- Rook Polynomials
- Disjoint Subboards
- The IE Formula for Rook Polynomials
- Computing Rook Polynomials

Excluded Positions

- Suppose we have n distinct items that must be placed by a bijection into n distinct positions. There are n! ways to do this.
- But now suppose that certain item-position pairs are unacceptable, and we want to count the bijections that meet these constraints.
- We can think of this as counting the perfect matchings in the bipartite graph whose edges represent the valid pairs.

A Five-Position Example

- In this example, seven of the 25 pairs are excluded. So of the 5! possible bijections, some are bad.
- We can let A_i be the set of maps that are bad in the i^{th} column. Then we are looking for the number of maps that are in none of the A_i's.

x	x			
	x			
		x	x	
			x	x

A Five-Position Example

- By the IE formula we want N - S_1 + S_2 - S_3 + S_4 - S_5, where S_k is the sum of the sizes of all k-way intersections of the A_i's.
- $|A_1|$ = 4! and $|A_2|$ = 2(4!). S_1 is the sum of one term of 4! for each excluded square, in this case 7(4!).

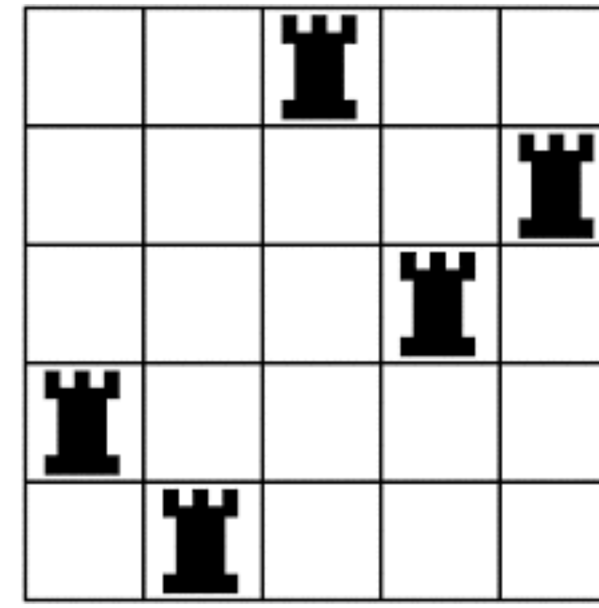
x	x			
	x			
		x	x	
			x	x

A Five-Position Example

- With some careful counting, we can get that $S_2 = 16(3!)$, $S_3 = 13(2!)$, $S_4 = 3(1!)$, and $S_5 = 0$.
- This makes the total $5! - 7(4!) + 16(3!) - 13(2!) + 3(1!) = 120 - 168 + 48 - 26 + 3 = 25$.
- Can we do this more systematically?

x	x			
	x			
		x	x	
			x	x

Placing Non-Capturing Rooks



- A chess rook can move any distance horizontally or vertically. A set of rooks on the board is **non-capturing** if no two are on the same row or the same column.
- Our whole problem was the number of ways to place n non-capturing rooks on the *valid* squares.
- It turns out to be useful to investigate placing them on the *invalid* squares, as we just saw.

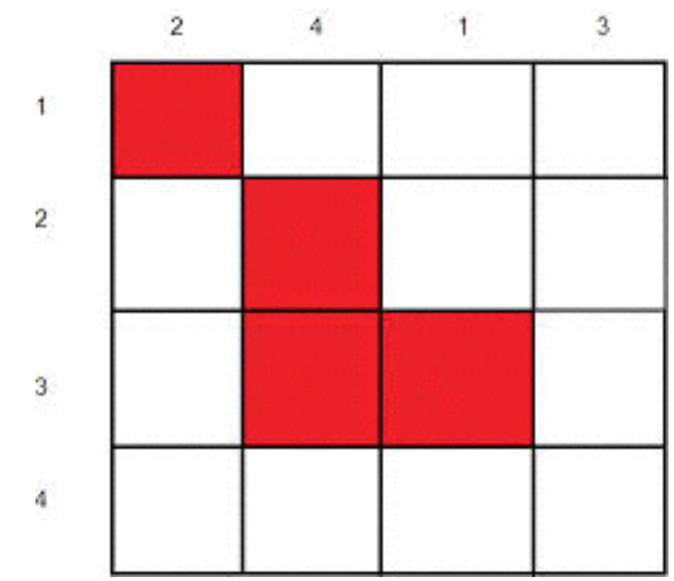
Rook Polynomials

- Let B be a board, a set of squares that is a subset of an n by n grid.
- We define the rook polynomial of B, called R(x, B), to be the polynomial in x whose x^k coefficient is $r_k(B)$, the number of ways to place k non-capturing rooks on the squares of B.
- This is a polynomial of degree at most n, but it is a property of the squares rather than of the number n.

Rook Polynomial Examples

- We always have $r_0(B) = 1$ and $r_1(B)$ equal to the number of squares in B.
- If B is contained within a single row or column, all r_i's for $i > 1$ are zero.
- An n by n square has $r_k(B) = k!C(n, k)^2$, as we saw on the practice second midterm.
- The n by n square without its main diagonal has $r_n(B) = D_n$, the derangement number.

Disjoint Subboards



- Why do we keep this information as a polynomial instead of just numbers?
- If a board has **disjoint subboards** as in this example, we can compute $R(x, B)$ as $R(x, B_1)R(x, B_2)$, just multiplying polynomials.
- Here B_1 is the top left square and B_2 is the rest. No square in one is in the row or column of a square in the other.

Disjoint Subboard Example

	x		x		
x					
x				x	
	x		x		
					x

- Permuting the rows or columns of a board does not change its rook polynomial.
- In this example we can permute to get a board that breaks into three disjoint subboards, so its polynomial is $(1+4x+2x^2)(1+3x+x^2)(1+x)$.

x	x				
x	x				
			x		
			x	x	
					x

IE Formula for Rook Polynomials

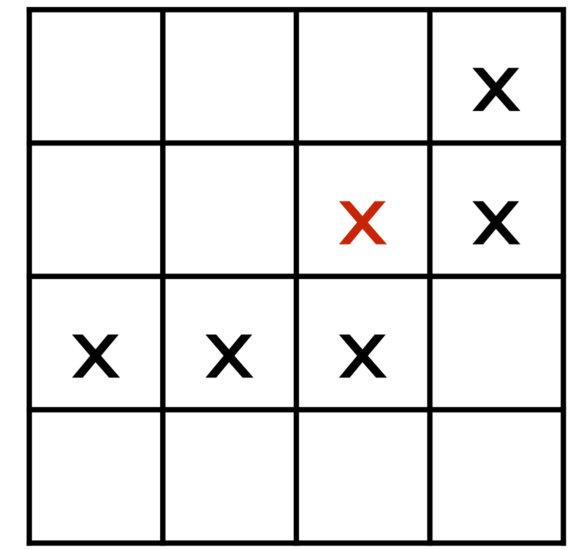
- Now we can apply the IE formula using rook polynomials to more quickly calculate the S_k's.
- S_k is just $r_k(B)(n-k)!$, where B is the board of excluded squares. So the total number of valid maps is $n! - r_1(B)(n-1)! + r_2(B)(n-2)! - \ldots + (-1)^n r_n(B)(n-n)!$.
- In our example, $R(x, B) = (1+4x+2x^2)(1+3x+x^2)(1+x) = 1+8x+22x^2+25x^3+12x^4+2x^5$ and our number is $6! - 8(5!) + 22(4!) - 25(3!) + 12(2!) - 2(1!) = 720 - 960 + 528 - 150 + 24 - 2 = 160$.

Computing Rook Polynomials

- This board does not break into disjoint pieces, but we can compute its rook polynomial.

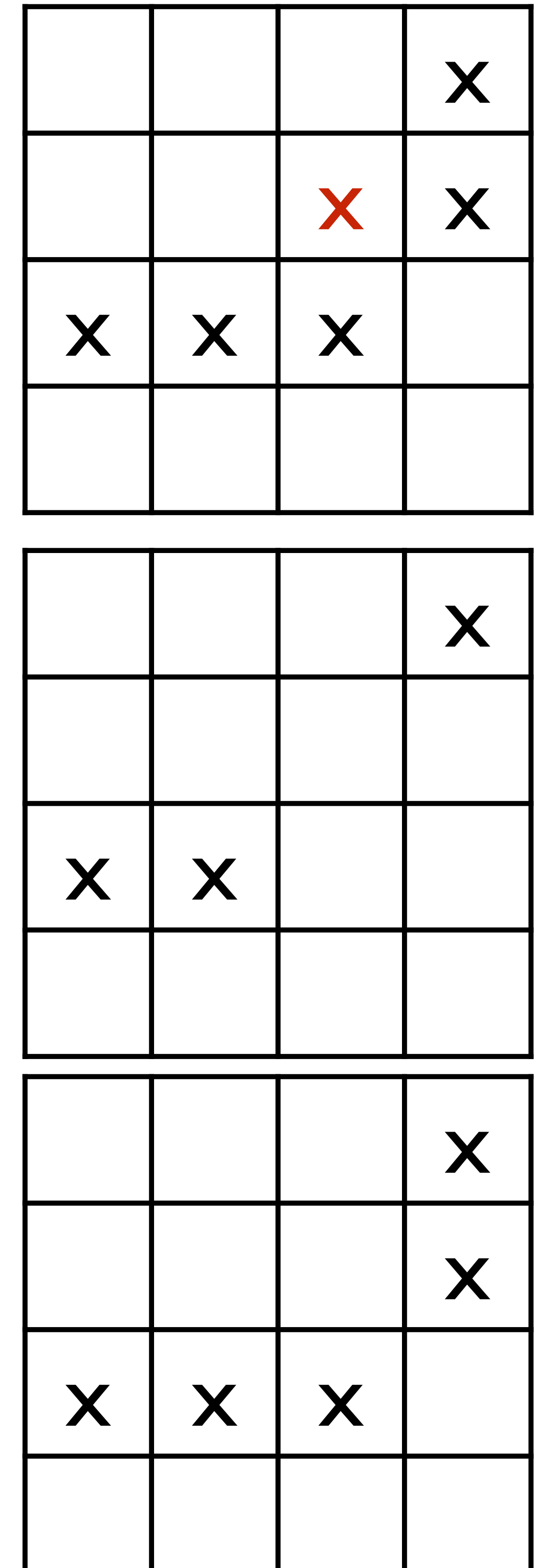
- Any set of rooks either contains the red-x square or it doesn't.

- The ones that do are counted by the rook polynomial of the board we get by removing the red square and all squares in its row or column, multiplied by x to account for the rook on the red-x square.



Computing Rook Polynomials

- The second board has polynomial $(1+x)(1+2x)$, so we get $(1+x)(2+x)x$.
- The sets that don't have the red-x square are counted by the rook polynomial of the third board, from just deleting the red x. This is $(1+3x)(1+2x)$.
- Similar decomposition can work for any board. It's faster with the right choice of red-x square.



One More Example

- Here when we decompose on the red x, the second board has polynomial $(1+2x)(1+2x)(1+x)$.
- The third has $(1+3x+x^2)^2(1+x)$, which we multiply by x to account for the red-x square.
- The sum works out to $1+8x+22x^2+25x^3+11x^4+x^5$.
- The number of maps is $720 - 8(120) + 22(24) - 25(6) + 11(2) - 1 = 159$.

x				
x	x			
	x	x		
		x	x	
				x

x				
x				
		x	x	
				x

x				
x	x			
		x		
		x	x	
				x