COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #26: Inclusion-Exclusion
(Tucker Sections 8.1, 8.2)
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Inclusion-Exclusion

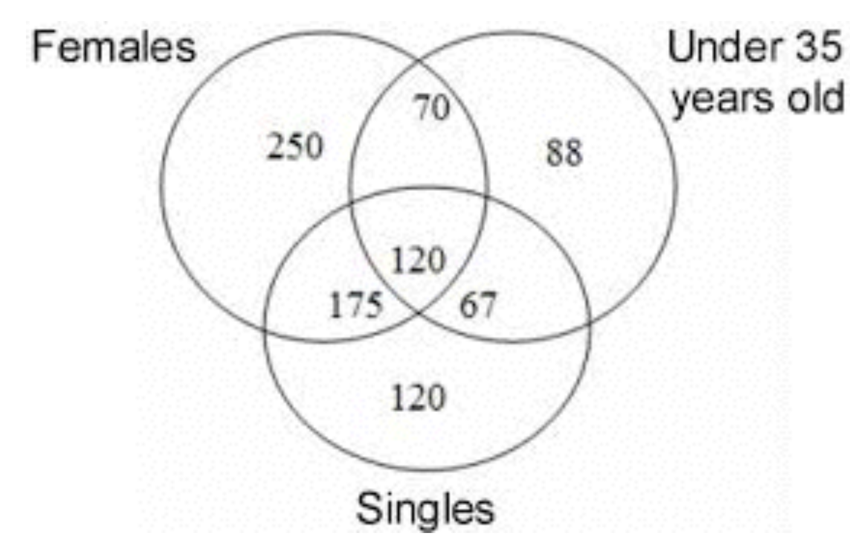
- Counting With Venn Diagrams
- Examples of Counting With Overlap
- The Inclusion-Exclusion Formula
- More Examples
- The Derangement Problem
- Chromatic Polynomials Again
- Number in Exactly or At Most m Sets

Counting With Venn Diagrams

- The single most fundamental rule for counting is the Sum Rule: if A and B are disjoint sets, $|A \cup B| = |A| + |B|$.
- Just after that is the Sum Rule With Overlap, $|A \cup B| = |A| + |B| - |A \cap B|$. We count both sets, and remove the double-counted items.
- We can expand this to more than two sets.

Counting With Venn Diagrams

- If I tell you that my total set is made up of 615 females, 345 young people, and 482 singles, that is not enough to compute its size.
- If in addition I say that there are 190 young females, 187 young singles, 295 single females, and 120 young single females, you can complete this diagram and compute the total of 890.



Restricted Arrangments

- Let's apply this to a counting problem. How many permutations of {0,…,9} have first digit greater than 1 and last digit less than 8?
- It's all of them (10!) minus those starting with 0 or 1 ($2 \cdot 9!$), minus those ending with 8 or 9 ($2 \cdot 9!$) *plus* those that both start with 0 or 1 and end with 8 or 9 ($2 \cdot 2 \cdot 8!$). The elements of the last set were subtracted twice and so must be added back in once.

Relatively Prime Numbers

- The set $\mathbb{Z}_{70}$ of integers mod 70 has a subset $\mathbb{Z}_{70}^*$ of elements relatively prime to 70.
- To count $\mathbb{Z}_{70}^*$, we start with 70, subtract the evens (35), multiples of 5 (14), and multiples of 7 (10), then add in the 7 multiples of 10, 5 multiples of 14, and 2 multiples of 35, then subtract 0 which has so far been subtracted twice and added in twice. The total of 70 - (35+14+10) + (7+5+2) -1 = (2-1)(5-1)(7-1) = 24, as we could also find from the Chinese Remainder Theorem.

More Examples

- How many n-digit sequences over {0,1,2} have at least one 0, at least one 1, and at least one 2? This isn't hard, but let's develop some useful notation. Let A_0, A_1, and A_2 be the sequences without 0, 1, and 2 respectively.
- The set we want to count is $A_0' \cap A_1' \cap A_2'$ where $'$ means complement because I can't do overlines.
- We compute it as $3^n - 3(2^n) + 3(1^n) - 0$.

More Examples

- If I have 100 students, 40 each are taking French, Latin, and German, and I am also given that 20 take only French, 20 only Latin, and 15 only German. 10 take both French and Latin.
- Is this enough to compute everything? We have eight possible combinations, represented in a three-set Venn diagram.
- We'll do this on the board, getting the conclusion that 15 students take none.

The Inclusion-Exclusion Formula

- In general, when we have a set of N elements and subsets $A_1,\ldots,A_n$, we will now write $N(A_1'A_2'\ldots A_n')$ to mean the number of elements not in any of the A_i's, omitting the ∩ symbols.
- We let $S_1 = |A_1| + \ldots + |A_n|$, S_2 be the sum of sizes of all the intersections $A_i \cap A_j$, S_3 be the sum of sizes of all 3-set intersections, and so on through all S_k to S_n, the size of the intersection of all n sets.

The Inclusion-Exclusion Formula

- With this notation, we have a theorem that the number $N(A_1'\ldots A_n')$ of elements in none of the sets is $N - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n$.
- To prove this, we look at an arbitrary element of the whole set and see how many times it is counted. If it is in m of the A_i's, we count it once in N, subtract it m times in S_1, add it back in C(m, 2) times, subtract it C(m, 3), etc.
- This sum is 1 if m = 0 and 0 otherwise.

Counting a Union

- Suppose I want the size of the union $A_1 \cup \ldots \cup A_n$. This is just N minus the number we just computed, which is thus $S_1 - S_2 + S_3 - \ldots + (-1)^{n+1} S_n$.
- This generalizes the formulas we have been using, $|A_1 \cup A_2| = S_1 - S_2$ and $|A_1 \cup A_2 \cup A_3| = S_1 - S_2 + S_3$.

Card Hands With No Voids

- Of the $C(52, 6)$ possible six-card hands from a standard deck, how many have at least one card of each suit?
- We let A_1 be the set of hands with no spades, A_2 those with no hearts, A_3 no diamonds, and A_4 no clubs. We can easily see that $|A_i| = C(39, 6)$, that $|A_i \cap A_j| = C(26, 6)$, that $|A_i \cap A_j \cap A_k| = C(13, 6)$, and that $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$.
- The IE formula gives us $C(52, 6) - 4C(39, 6) + 6C(26, 6) - 4C(13, 6) + 0$.

Upper Bounds on Solutions

- We earlier used generating functions to attack problems like the following: How many solutions to $x_1+\ldots+x_6 = 20$ have $0 \leq x_i \leq 8$?
- We let A_i be the set of non-negative solutions to this sum that have $x_i > 8$.
- We know N = C(20+6-1, 20), that $|A_i|$ = C(11+6-1, 11), and that $|A_i \cap A_j|$ = (2+6-1, 2). (The larger intersections are empty.) So our answer is C(25, 20) - 6C(15, 11) + 15C(7, 2).

The Derangement Problem

- If n people each check a hat and the hats are returned to them randomly, what is the probability that *no one* gets their own hat?
- We let N = n! be the set of all permutations, and let A_i be the ones where person i gets their own hat back.
- It is easy to see that $|A_i| = (n-1)!$, and that the intersection of k A_i's has size $(n-k)!$.

The Derangement Problem

- The number S_k is thus $C(n, k)(n-k)! = n!/k!$.
- Applying the IE formula, we compute the number of permutations not in any A_i as n! times the sum for k from 0 to n of $(-1)^k/k!$.
- This last sum is the sum of the first n+1 terms of the power series for $e^{-1} = 1/e$, and this number is the probability that no one gets their own hat.

More on Derangements

- The number D_n of derangements is also given by the recurrence $D_n = nD_{n-1} + (-1)^n$ for $n \geq 2$, with $D_0 = 1$ and $D_1 = 0$.
- This recurrence can be derived by arithmetic from the more natural $D_n = (n-1)(D_{n-1} + D_{n-2})$, which comes from seeing where the first item goes and whether it is in an orbit of size 2.
- This recurrence can also be used to get the EGF for D_n, which is $e^{-x}/(1-x)$.

Chromatic Polynomials Again

- Let G be a graph with vertices x_1, x_2, x_3, and x_4, and five edges (all but x_2x_4). How many legal n-colorings does this graph have?
- We let N be the total number of colorings of the four vertices, n^4, and let $A_1,\ldots,A_5$ be the sets of colorings that fail the test for each of the five edges.
- We have $|A_i| = n^3$ for each edge, and $|A_i \cap A_j| = n^2$ for each pair of edges.

Chromatic Polynomials Again

- What about the three-way intersections? Of the ten sets of three edges, two form triangles and can each have a common color in n^2 ways, while the other eight involve all four vertices and allow only n colorings. The four-way and five-way intersections each allow n colorings.
- The chromatic polynomial, our answer, is thus $n^4 - 5n^3 + 10n^2 - [2n^2 + 8n] + 5n - n$ which is $n^4 - 5n^3 + 8n^2 -4n$.

Number in m or $\geq$ m Sets

- Again let us have a set of N elements, with n subsets $A_1,\ldots,A_n$. How many of the elements are in exactly m of the subsets? Call this N_m.
- We can start with S_m, the sum of the sizes of all m-way intersections. Every element in exactly m sets will be in exactly one of these m-way intersections.
- But the elements in more than m subsets will also be counted multiple times in S_m.

Number in m or ≥ m Sets

- The correct formula for N_m is similar to the IE formula: $N_m = S_m - C(m+1, m)S_{m+1} + C(m+2, m)S_{m+2} + \ldots + (-1)^{n-m}C(n, m)S_n$.
- The similar formula for N_m^*, the number of elements in at least m sets, is $N_m^* = S_m - C(m, m-1)S_{m+1} + C(m+1, m-1)S_{m+2} + \ldots + (-1)^{n-m}C(n-1, m-1)S_n$.
- We verify each of these formulas by showing that each element we don't want is counted a net of zero times.

One More Example

- Consider strings of length 4 over {0, 1, 2} with exactly two 1's. If A_i is the set of strings with a 1 in position i, we want the number N_2. By the formula this is S_2 - $C(3, 2)S_3$ + $C(4, 2)S_4$. (Tucker has some typos here.)
- Since $S_2 = C(4, 2)3^2 = 54$, $S_3 = C(4, 3)3^1 = 12$, and $S_4 = 1$, we have $54 - 3 \cdot 12 + 6 \cdot 1 = 24$.
- Similarly $N_2^* = S_2 - C(2, 1)S_3 + C(3, 1)S_4 = 54 - 2 \cdot 12 + 3 \cdot 1 = 33$.