COMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #25: Recurrences and Generating Functions
(Tucker Section 7.5)

David Mix Barrington

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Recurrences and GF's

- Functional Equations
- The Pizza Problem Again
- Fibonacci Again
- Selection Without Repetition
- Catalan Numbers Again
- Simultaneous Recurrences
- The Method of Partial Fractions

Functional Equations

- If I have a function $a_0, a_1, \ldots$ defined by a recurrence, it has an associated GF $g(x) = a_0 + a_1x + a_2x^2 + \ldots$, and sometimes we can use the recurrence to determine the GF.
- If we can relate $g(x)$ to itself and to other functions of x in a **functional equation**, we may be able to solve this equation to determine $g(x)$

Functional Equation Example

- Suppose I can determine that $g(x) = x^2g(x) - 2x$. Then I can treat $g(x)$ as a single variable y, giving $y = x^2y - 2x$, and then solve for y treating functions of x alone as constants, getting $y = -2x/(1-x^2)$.
- Given an equation like $(1-x^2)[g(x)]^2 - 4xg(x) + 4x^2 = 0$, we can apply the quadratic formula. We have $ay^2 + by + c = 0$, with $a = 1-x^2$, $b = -4x$, and $c = 4x^2$.
- This solves to $y = (4x \pm 4x^2)/2(1-x^2)$, two solutions from which we pick one matching a_0.

The Pizza Problem Again

- Remember that if a_n is the number of pieces we can make by n straight cuts of a convex pizza, we had $a_1 = 1$ and $a_n = a_{n-1} + n$.
- For every n with $n \geq 1$, we have $a_n x^n = a_{n-1} x^{n-1} + nx^n$. Summing these terms, we get $g(x) - a_0 = \sum_1^\infty (a_{n-1}x + nx^n) = xg(x) + x/(1-x^2)$. (Remember that $1/(1-x)^2 = 1 + 2x + 3x^2 + \ldots$)
- So $g(x) - 1 = xg(x) + x/(1-x)^2$, $g(x)(1-x) = 1 + x/(1-x)^2$, and $g(x) = 1/(1-x) + x/(1-x)^3$. This solves to $a_n = 1 + C(n+1, 2)$ as we had before.

Fibonacci Again

- Let's now solve the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$, with $a_0 = a_1 = 1$. We let $g(x)$ be the sum for all n of $a_n x^n$, and then $g(x) - a_0 - a_1 x$ is the same sum for all n with $n \geq 2$, where the recurrence holds.
- We get $g(x) - a_0 - a_1 x = \sum_2^\infty (a_{n-1}x^n + a_{n-2}x^n) = x[g(x) - a_0] + x^2 g(x)$.
- This gives $g(x)(1-x-x^2) = 1$ or $g(x) = 1/(1-x-x^2)$, which we can solve by the quadratic formula.

Fibonacci Again

- The roots of $1-x-x^2 = 0$ are $\alpha_1 = (1+\sqrt{5})/2$ and $\alpha_2 = (1-\sqrt{5})/2$, so that the denominator of $g(x)$ factors into $(1-\alpha_1 x)(1-\alpha_2 x)$.
- By the **method of partial fractions**, we write $1/(1-\alpha_1 x)(1-\alpha_2 x)$ as $y/(1-\alpha_1 x) + z/(1-\alpha_2)$ and solve for y and z to get the values $y = \alpha_1/\sqrt{5}$ and $z = -\alpha_2/\sqrt{5}$.
- Now $1/(1-\alpha_1 x)$ is the GF for $1+\alpha_1 x+\alpha_1{}^2 x^2+\ldots$, and $1/(1-\alpha_2 x)$ is the GF for $1+\alpha_2 x+\alpha_2 x^2+\ldots$

Fibonacci Again

- This means that $g(x) = y/(1-\alpha_1 x) + z/(1-\alpha_2 x)$ is the GF for $a_n = y\alpha_1^n + z\alpha_2^n$, just as we found before.
- Of course, to find the value of a_{10} we would be much better off calculating $a_2, a_3, \ldots, a_{10}$ in order using the recurrence, rather than evaluating the GF coefficient.
- We can use similar methods with any linear recurrence.

Method of Partial Fractions

- Let's take another look at using partial fractions to solve general homogeneous linear recurrences.
- We know that over the complex numbers, a degree-r polynomial g(x) factors into the product of r linear polynomials, with some perhaps multiple.
- If g(x) = 0, any polynomial at all is equal to one of degree at most r-1, by long division.

Method of Partial Fractions

- Suppose that $g(x) = (1-\alpha)(1-\beta)(1-\gamma)^2$. Consider any polynomial of the form $A/(1-\alpha) + B/(1-\beta) + (Cx+D)/(1-\gamma)^2$.
- By taking a common denominator, we can show that this polynomial is equal to $f(x)/g(x)$, where $f(x)$ has degree at most $r-1$.
- And given any such $f(x)$, we can find A, B, C, and D to put it in the other form. This explains, for example, the $An+B$ term in our general solution when we have a double root.

Selection Without Repetition

- In the spirit of solving more known problems in new ways, let's look again at the number of ways to choose k objects from a set of n objects, without repetition.
- Consider a family of GF's $g_0, g_1, g_2, \ldots$ with $g_n(x) = a_{n,0} + a_{n,1}x + a_{n,2}x^2 + \ldots$ for each n. We'll let $a_{n,k}$ be our desired number.
- We know that these coefficients satisfy the recurrence rule $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$, with initial conditions $a_{n,0} = 1$ and $a_{0,k} = 0$ for $k > 0$.

Selection Without Repetition

- For each n, $g_n(x) - 1 = \sum_1^n (a_{n-1,k}x^k + a_{n-1,k-1}x^k) = g_{n-1}(x) - 1 + xg_{n-1}(x)$.
- This yields a functional equation $g_n(x) = (1+x)g_{n-1}(x)$, which solves to $g_n(x) = (1+x)^n$ with the initial condition $g_0(x) = 1$.
- So from the recurrence, we get a generating function that we recognize by the binomial theorem, so we know that $a_{n,k} = C(n, k)$.

Catalan Numbers Again

- Placing parentheses to multiply n numbers gave us the Catalan recurrence relation, with $a_n = a_1 a_{n-1} + \ldots + a_{n-1} a_1$, $a_0 = 0$, and $a_1 = 1$.
- Why is this? The first left parenthesis and its matching right parenthesis enclose some number i of the n numbers. For each i, there are a_i ways to group those first i numbers and a_{n-i} ways to group the last n-i.

Catalan Numbers Again

- The parenthesizing sequence starts out $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 5$, $a_5 = 14$, and $a_6 = 42$.
- The Catalan recurrence describes a number of combinatorial problems, with varying initial conditions.
- If I have an n-node rooted binary tree with i nodes in its left subtree, it has n-i-1 nodes in its right subtree. So if t_n is the number of n-node trees, we have that $t_n = t_0 t_{n-1} + \ldots + t_{n-1} t_0$, with $t_0 = t_1 = 1$, giving $t_2 = 2$, $t_3 = 5$,… which is the same sequence shifted by one.

Catalan Numbers Again

- If $g(x) = a_0+a_1x+a_2x^2+\ldots$, the RHS is the coefficient of x^n in $g(x)g(x)$ for $n \geq 2$, and we get $g(x) - a_1x - a_0 = g(x) - x = [g(x)]^2$.
- Solving this quadratic equation gives $g(x) = (1 \pm \sqrt{(1-4x)})/2$. For the parenthesizing sequence, we want to make $g(0) = 0$, so we choose $(1-\sqrt{(1-4x)})/2$.

Generalized Binomial Theorem

- How can get coefficients for a GF like $\sqrt{(1-4x)}$?
- This involves a generalization of the binomial theorem, involving a generalization of binomial coefficients.
- We can still define $(1+y)^q$, where q is any real number (not necessarily an integer), as the sum of $C(q, n)y^n$, where $C(q, n)$ must be defined.
- We let $c(q, n)$ be $q(q-1)(q-2)\ldots(q-n+1)/n!$, just as for integers.

Generalized Binomial Theorem

- What does this tell us when $q = 1/2$? We get $C(1/2, 0) = 1$, $C(1/2, 1) = 1/2$, $C(1/2, 2) = (1/2)(-1/2)/2! = -1/8$, and in general $C(1/2, n) = 1(-1)(-3)(-5)\ldots(-(2n-3))/2^n n!$.
- This lets us evaluate $(1-4x)^{1/2}$. We get the sum over all n of $C(1/2, n)(-4)^n = -1(1)(3)\ldots(2n-3)2^n/n!$.
- Some fooling around with powers of to gets us from this to the fact that the n^{th} Catalan number is $(1/n)C(2n-2, n-1)$. I'll omit the details here.

Simultaneous Recurrences

- Example 5 of Tucker's section 7.5 attacks a system of simultaneous recurrences: $a_n = a_{n-1} + b_{n-1} + c_{n-1}$, $b_n = 3^{n-1} - c_{n-1}$, and $c_n = 3^{n-1} - b_{n-1}$.
- These arose from the example of ternary strings of length n, where a_n is the number with an even number of 0's and an even number of 1's, b_n the number with even 0's and odd 1's, and c_n the number with odd 0's and even 1's. (The fourth case is $3^n-a_n-b_n-c_n$.)

Simultaneous Recurrences

- We also have initial conditions $a_0 = 1$, $b_0 = 0$, and $c_0 = 0$.
- Let A(x), B(x), and C(x) be the GF's for these three sequences.
- Tucker goes through a derivation where he expresses each of these GF's as a function of the others, for example $A(x) - 1 = xA(x) + xB(x) + xC(x)$ and $B(x) - 1 = x/(1-3x) - xC(x)$. Having each of B and C in terms of the other lets him solve for those two, then find A.