COMPSCI 575/MATH 513 T

Combinatorics and Graph Theory

Lecture #22: Exponential Generating Functions
(Tucker Section 6.4)
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Exponential GF's

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A Motivating Problem

- Exercise 5.2.38 in Tucker (on HW#4) asks about ten people who order sandwiches at a deli. Eight always order the same thing (four tuna, two roast beef, two chicked) and the other two vary their order between the three choices.

- Part (b) asks how different total sandwich orders are possible, and there are six: 6T2R2C, 5T3R2C, 5T2R3C, 4T4R2C, 4T3R3C, and 4T2R4C. This is C(2+3-1,2) because we choose a multiset of size 2 from {T, R, C} for the variable orders.

A Motivating Problem

- But part (a) asks how many possible sequences of sandwiches, such as TTRRTCTCCT, are possible. That is, how many total arrangements can be made of these six multisets?
- 6T2R2C has P(10; 6, 2, 2) = C(10, 6)C(4, 2) = 1260, while 5T3R2C has P(10; 5, 3, 2) = C(10, 5)C(5, 3) = 2520, as does 5T2R3C. 4T4R2C and 4T2R4C each have 3150, and 4T3R3C has 4200, for a total of 16800.

A Motivating Problem

- With GF's, we can solve part (b) easily because the answer of 6 is the x^{10} coefficient of $x^4/(1-x)$ times $x^2/(1-x)$ times $x^2/(1-x)$, which is the x^2 coefficient of $1/(1-x)^3$ or C(2+3-1, 2).
- But ordinary GF's don't appear to help us with part (a). Each of the six terms $x^6x^2x^2$, $x^5x^2x^3$, $x^5x^2x^3$, $x^4x^4x^2$, $x^4x^3x^3$, and $x^4x^2x^4$ contributes a different number of arrangements to the total number for the ten elements.

Defining Exponential GF's

- This leads us to a new definition, for a new kind of GF for a sequence $a_0, a_1, a_2, \ldots$ called an **exponential generating function** or **EGF**.
- The EGF for $\{a_r\}$ is $a_0/0! + a_1x/1! + a_2x^2/2! + a_3x^3/3! + \ldots$, where we divide each x^r term of the ordinary GF by $r!$.
- Thus the EGF for 1, 1, 1,… is $1 + x + x^2/2 + x^3/6 + x^4/24 + \ldots$, which you may recognize as the Taylor expansion of the function e^x.

EGF's for the Sandwiches

- Let's see what happens if instead of multiplying the GF's for our three sandwich flavors, we multiply the EGF's.
- The sequence of ways to have r tuna sandwiches is 0, 0, 0, 0, 1, 1, 1,…, and thus the EGF is $x^4/4! + x^5/5! + x^6/6! + \ldots$
- The other two EGF's are both $x^2/2! + x^3/3!/ + x^4/4! + \ldots$

EGF's for the Sandwiches

- Multiplying $(x^4/4!+x^5/5!+\ldots)(x^2/2!+x^3/3!+\ldots)^2$ gives us a power series whose x^{10} coefficient is 1/6!2!2! + 1/5!2!3! + 1/5!3!2! + 1/4!4!2! + 1/4!3!3! + 1/4!2!4!.
- This new power series is the EGF for a sequence $b_0, b_1, b_2, \ldots$ where b_{10} is exactly the sum of terms above, times 10!. Hence this sum is exactly the sum of P(10; 6, 2, 2) and the permutation numbers from the other five partitions.

Exponential GF for P(n, r)

- Suppose now that we want an EGF for the number of length-r arrangements of n objects, without repetition.
- The ordinary GF is $(1+x)^n$, since each object is there either 0 times or 1 time. This is also the EGF for this sequence of choices, since dividing the terms by 0! or 1! has no effect.
- For what sequence of numbers is $(1+x)^n$ the EGF? We have $a_r/r! = C(n, r)$, so $a_r = P(n, r)$.

Arranging Objects

- For another example, let's have four types of objects and pick from two to five of each type.
- The EGF for each type is $(x^2/2!+x^3/3!+x^4/4!+x^5/5!)$, so the entire EGF is $(x^2/2!+\ldots+x^5/5!)^4$.
- If we view this as the EGF for $a_1,a_2,\ldots$, then a_r is the sum of terms of the form $r!/e_1!e_2!e_3!e_4!$ for all sums of the form $e_1+e_2+e_3+e_4 = r$.
- And this is exactly the number of arrangements of r objects chosen from the four types in this way.

Relating Exponential GF's to e^x

- Unfortunately, EGF's are much more difficult to compute with than ordinary GF's.
- We have the Taylor series for e^x, and more generally $e^{nx} = 1+nx+n^2x^2/2!+\dots$ is the EGF for the sequence $1, n, n^2,\dots$
- But given, say, $x^2/2!+x^3/3!+x^4/4!+\dots$, we can't factor out an x^2 as we did with the ordinary GF. The best we can do for this EGF is to write it as $e^x - 1 - x$.

Even and Odd Terms

- There are two more useful identities for EGF's. We know that e^x can be written as $1+x+x^2/2!+x^3/3!+\ldots$, and e^{-x} as $1-x+x^2/2!-x^3/3!+\ldots$
- This gives us that $(e^x+e^{-x})/2 = 1+x^2/2!+x^4/4!+\ldots$ and that $(e^x-e^{-x})/2 = x+x^3/3!+x^5/5!+\ldots$
- (These might remind you of the Taylor series for trigonometric functions, which we'd get by plugging in some i's here and there.)

More Examples

- Here's an easy example first. Let's use EGF's to solve our first counting problem, the number of ways to choose r objects from n types with unlimited repetition.
- The EGF for a single type is just $e^x = 1+x+x^2/2!+\ldots$, so the EGF for n types is the product $(e^x)^n = e^{nx} = 1+nx+n^2x^2/2!+\ldots$, and this is just the EGF for the sequence $1, n, n^2,\ldots$
- Our answer is thus just the familiar n^r.

More Examples

- Now let's put 25 distinct people into three distinct rooms, with at least one person in each room.
- The EGF for each room is $x+x^2/2!+x^3/3!+\ldots$, which is also e^x-1, so the total EGF is $(e^x-1)^3 = e^{3x} - 3e^{2x} + 3e^x - 1$.
- The x^{25} coefficient of e^{3x} is $3^{25}/25!$. The x^{25} coefficient of $-3e^{2x}$ is $-3(2^{25})/25!$. That of $3e^x$ is just $3/25!$, and that of 1 is just 0. So our final answer, 25! times the coefficient, is $3^{25} - 3(2^{25}) + 3$. Can you explain this answer combinatorially?

More Examples

- Now let's look at strings of length r over {a, b, c, d} with an even number of a's and an odd number of b's.
- The EGF is $(1+x^2/2+\ldots)(x+x^3/3!+\ldots)(1+x+x^2/2)^2$, which by our identities is $(e^x+e^{-x})/2$ times $(e^x-e^{-x})/2$ times e^{2x}.
- This is $(1/4)(e^{2x}-e^{-2x})e^{2x} = (e^{4x}-1)/4$. That makes the number of strings of length r equal to $4^r/4 = 4^{r-1}$. Can you explain this?

Stirling Numbers Again

- Remember that we earlier looked at the number of ways to put r distinct objects into n distinct boxes with at least one in each box.
- The EGF for each box is e^x-1, so our overall EGF is $(e^x-1)^n$. For any fixed n, we can expand this to the sum over k of $(-1)^{n-k}C(n, k)e^{kx}$
- This makes the number $s_{n,r}$ of arrangements equal to the sum over all k of $(-1)^{n-k}C(n, k)k^r$, since e^{kx} is the sum over r of $k^r/r!$.

Stirling Numbers Again

- This quantity $s_{n,r}$ is the number of maps of r distinct objects into n distinct boxes.
- The Stirling number of the second kind is the number of maps into identical boxes, which is just $s_{n,r}/n!$, which we earlier called $S(n, r)$.
- We'll see the Stirling numbers of the first kind later. There $s(n, r)$ is the number of permutations of n elements that have r **orbits** of elements. (This *isn't* what we just called $s_{n,r}$, but there are only so many letters…)