COMPSCI 575/MATH 513 T

Combinatorics and Graph Theory

Lecture #21: Partitions
(Tucker Section 6.3)
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Partitions

- Defining Partitions
- A GF for the Number of Partitions
- Wait — Infinite Products?
- Partitions into Distinct Integers
- Choosing Stamps
- Integers as Powers of Two
- Ferrers and Young Diagrams

Defining Partitions

- A **partition** of a set S is a collection of pairwise disjoint subsets that union to S.
- Similarly, a **partition** of a non-negative integer r is a collection of positive integers that sum to r.
- Equivalently, a partition of r is a mapping from r identical objects into some number k of identical boxes.
- We'll let $p_k(r)$ be the number of partitions into k boxes, and p(r) be the total number of partitions.

Partition Examples

- We can represent a partition by a sum where the terms are in non-decreasing order.
- Of course 0 has one partition the empty one, and 1 can only be divided into itself.
- 2 can be 1+1 or 2, and 3 can be 1+1+1, 1+2, or 3. 4 can be 1+1+1+1, 1+1+2, 1+3, 2+2, or 4, and 5 can be 1+1+1+1+1, 1+1+1+2, 1+1+3, 1+2+2, 1+4, 2+3, or 5. Note that the latter list is the set of poker hands, except for straights and flushes.

Partition Examples

r	1	2	3	4	5	6	7
$p_1(r)$	1	1	1	1	1	1	1
$p_2(r)$	0	1	1	2	2	3	3
$p_3(r)$	0	0	1	1	2	3	4
$p_4(r)$	0	0	0	1	1	2	3
$p_5(r)$	0	0	0	0	1	1	2
$p_6(r)$	0	0	0	0	0	1	1
$p_7(r)$	0	0	0	0	0	0	1
$p(r)$	1	2	3	5	7	11	15

- Here are some values of $p_n(r)$ and $p(r)$.
- The 4 for $p_3(7)$ counts 115, 124, 133, and 233.

GF for the Partition Numbers

- The partition numbers don't appear to obey any rule that can be characterized by a formula, though they have been calculated for rather large r (p(10000) has 107 digits).
- But we can express the entire function p(r) by a GF. We can characterize a partition by how many 1's, 2's, 3's, etc. it has. If e_k is the number of k's, we have that $e_1 + 2e_2 + 3e_3 + \ldots + re_r = r$. We are picking r objects in the form of e_1 singletons, e_2 pairs, e_3 triplets, etc.

GF for the Partition Numbers

- Look at the function $(1+x+x^2+x^3+\ldots)$ times $(1+x^2+x^4+x^6+\ldots)$ times $(1+x^3+x^6+x^9+\ldots)$ and so on, all the way out to $(1+x^r+x^{2r}+x^{3r}+\ldots)$.
- There will be one term in this product for every choice of $e_1,\ldots, e_r$, and the degree of that term will be $e_1+e_2+\ldots+e_r$.
- So the GF for the entire partition number, with any value of r, is $1/(1-x)(1-x^2)(1-x^3)\ldots$, a product of infinitely many power series.

Wait—Infinite Products?

- But can we even do that? We showed last time that any *finite* product of power series with integer coefficients is itself a series with integer coefficients. But is this true for *infinite* products?

- Clearly it's not always the case. If I multiply together an infinite number of copies of the series $1/(1-x) = 1+x+x^2+\ldots$, I get an infinite number of linear terms by taking one x and taking 1's in all the other series.

Wait—Infinite Products?

- If $f_1, f_2, f_3, \ldots$ are a sequence of power series, I can define g_n to be the product of the first n series in the sequence. These series g_n all definitely exist, but our question is whether they **converge** to a single series that could be defined as our infinite product.
- Convergence requires a metric, or at least a topology, on the space of all power series. We can take the length of f to be 2^{-k}, where k is the number of 0 terms at the start of f.

Wait—Infinite Products?

- The sequence of g_n's converges to h if for any positive real number δ, the length of $h - g_n$ is less than δ for sufficiently large n. With our metric, this means that for any k, h agrees with g_n on the first k terms for sufficiently large n.
- For the product we're considering, we get each g_{n+1} by multiplying g_n by $1+x^{n+1}+\ldots$, which adds only terms of degree at least n+1.
- So we will have convergence in this case.

Is This Useful at All?

- You will notice that Tucker gives no examples of computing partition numbers from the generating function.
- That's because, I think, it would be tedious for hand calculation. If we wanted the x^r coefficient of the GF, we could calculate it even though it is an infinite product, because only the terms $1(1\text{-}x)$, $1/(1\text{-}x^2)$,…, $1/(1\text{-}x^r)$ would affect the coefficient we want.
- We could do this in time polynomial in r.

Partitions Into Distinct Integers

- What we can do is mimic the derivation of this GF to get GF's for similar problems.
- What about the number of ways to write r as a sum of *distinct* integers?
- We truncate each term $1+x^k+x^{2k}+\ldots$ to just $1+x^k$, so that our GF is the infinite product $(1+x)(1+x^2)(1+x^3)(1+x^4)\ldots$
- The product starts out $1+x+x^2+2x^3+2x^4+3x^5$: the sums for r=5 are 1+4, 2+3, and 5.

Choosing Stamps

- How many ways are there to make up r cents from 2-cent, 3-cent, and 5-cent stamps?
- The GF for this problem is just $(1+x^2+x^4+\ldots)(1+x^3+x^6+\ldots)(1+x^5+x^{10}+\ldots)$, or equivalently $1/(1-x^2)(1-x^3)(1-x^5)$.
- To find the x^r coefficient, we can multiply the three polynomials we get by truncating the series to the first r+1 terms.

Integers as Powers of Two

- Of course we know that every positive integer can be written in a unique way as a sum of distinct powers of two. This is easy to prove by induction, but we can get an interesting alternate proof using generating functions.

- Just as in the problem before last, the GF for the number of ways to write r as a sum of *distinct* powers of two is just $(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots$, an infinite product.

Distinct Powers of Two

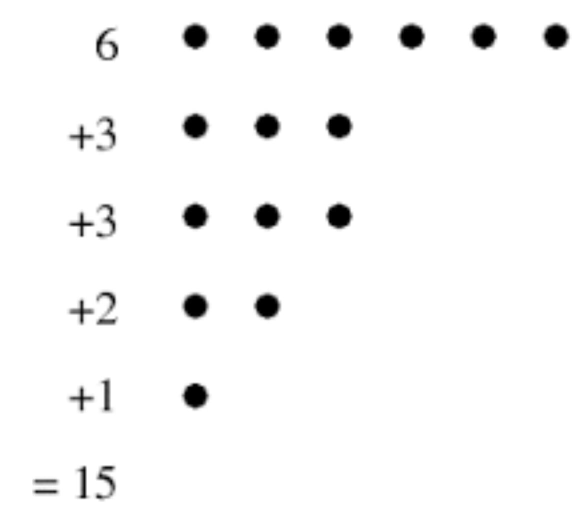
- Let g(x) be this product $(1+x)(1+x^2)(1+x^4)\ldots$
- If we compute $(1-x)g(x)$, we can multiply the first two terms to get $1-x^2$, then multiply that by the next term to get $1-x^4$, then multiply that by $1+x^4$ to get $1-x^8$, and so forth.
- The claim is that this infinite product is exactly 1. How do we argue that? We show that the partial products converge to 1, because each one differs from 1 by a "smaller" series, one with higher-degree terms.

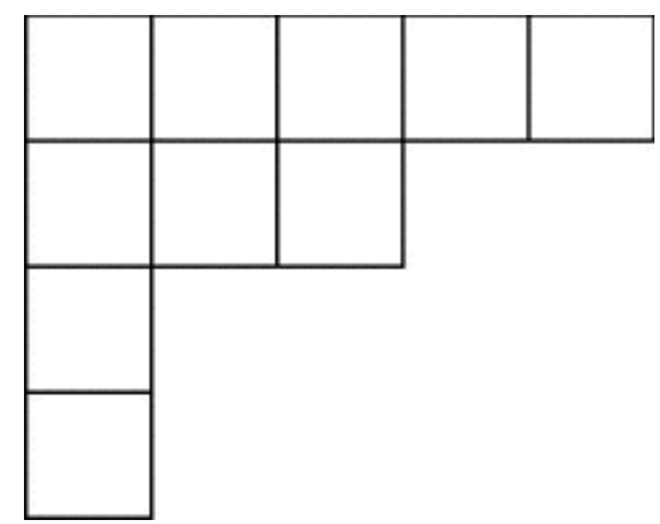
Distinct Powers of Two

- If we believe that $(1-x)g(x) = 1$, it follows that $g(x) = 1/(1-x) = 1+x+x^2+\ldots$, and thus that the number of ways to write *any* r as a sum of distinct powers of two is exactly 1.
- Tucker is pretty casual about the "…" in the infinite product, but thinking of the infinite product as the limit of its partial products (if that limit exists) takes care of the problem.

Ferrers and Young Diagrams

- A **Ferrers diagram** for a partition has a row of dots for each piece, in descending order of size.
- A Young diagram is similar, with squares instead of dots.





Young diagram for 5+3+1+1

Ferrers and Young Diagrams

- Given any Ferrers diagram, we can form its **conjugate** by transposing the rows and columns to get a different partition of the same number.
- In this way partitions of r into exactly k pieces are in 1-1 correspondence with partitions whose largest piece is size exactly k.