CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #18: Binomial Coefficient Identities
(Tucker Section 5.5)

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Binomial Coefficient Identities

- The Binomial Theorem
- The Two Biggest Identities
- Paths in Manhattan
- Seven More Identities
- Attacking Sums
- Proofs by Substitution

Binomial Coefficient Identities

- Today we will be working almost entirely with **binomial coefficients**, the answers to one of our basic combinatorial problems.
- The number C(n, r), also called "n choose r", is the number of r-element subsets of an n-element set. It is also the number of binary strings with r 0's and n-r 1's.
- The usual notation for C(n, r) has the n above the r inside parentheses, but that is hard to create with this editor so I'll use "C(n, r)".

The Binomial Theorem

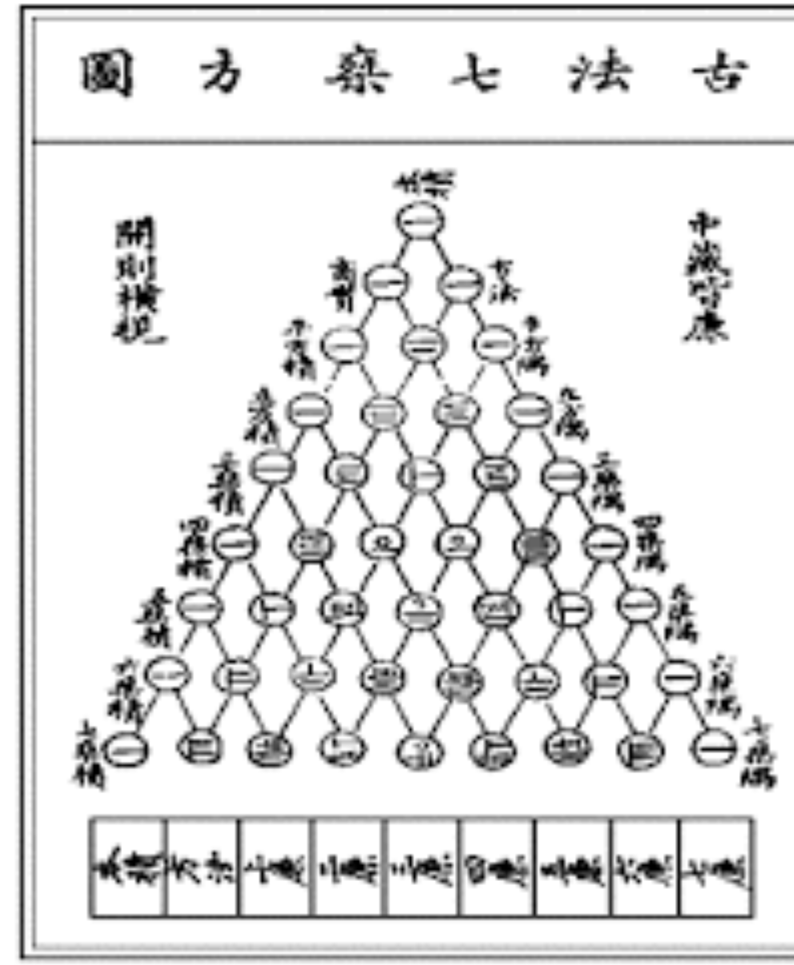
- Binomial coefficients get their name from the **Binomial Theorem**, which says that $(x+y)^n$ is equal to the sum, for i from 0 to n, of the term $C(n, i)x^iy^{n-i}$.
- Raising x+y to the n^{th} power gives us a sum of 2^n terms, one for every string of x's and y's of length n. If we collect the terms with i x's and n-1 y's, we get exactly C(n, i) of them.
- We can also prove the Binomial Theorem by induction, once we have Pascal's Identity.

The Two Biggest Identities

- It is easy to prove that C(n, k) = C(n, n-k), from either the string or subset points of view. We can swap 0's and 1's in the binary strings, or pair each subset with its complement.
- **Pascal's Identity** says that C(n, k) = C(n-1, k) + C(n-1, k-1). Again a combinatorial proof is easy: look at forming a string with k 0's and n-k 1's by appending a letter, or at forming a size-k subset of an n-element set by adding an element to an n-1 element set.

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1



It is convenient to represent the values of C(n, k) in a triangular table, which is symmetric and in which each entry is the sum of the two above it.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

Paths in Manhattan

- Shortest paths from (0, 0) to (x, y) in a Manhattan grid can be represented by sequences of N's and E's, with x E's and y N's. Clearly there are C(x+y, y) of these, and so exactly that many paths.
- We can prove Pascal's Identity again by noting that any path to (n-k, k) must pass through (n-k-1, k-1) or (n-k-1, k) but not both. So the C(n, k) paths to (n-k, k) are in bijection with the union of sets of C(n-1, k-1) and C(n-1, k) paths.

Seven More Identities

- $C(n, 0) + \ldots + C(n, n) = 2^n$
- $C(n, 0) + C(n+1, 1) + \ldots + C(n+r, r) = C(n+r+1, r)$
- $C(r, r) + C(r+1, r) + \ldots + (n, r) = C(n+1, r+1)$
- $C(n, 0)^2 + C(n, 1)^2 + \ldots + C(n, n)^2 = C(2n, n)$
- Sum k=0 to r of C(m, k)C(n, r-k) = C(m+n, r)
- Sum k=0 to m of C(m, k)C(n, r+k) = C(m+n, m+r)
- Sum k=n-s to m-r of C(m-k, r)C(n+k, s) = C(m+n+1, r+s+1)

Some Proofs

- All of these identities are easy to prove by counting Manhattan paths.
- C(r, r) + C(r+1, r) + … + (n, r) = C(n+1, r+1)
- A path from (0, 0) to (n-r, r+1) must at some point go from (k, r) to (k, r+1) for some k. There are exactly C(r+k, r) ways to get to the point (k, r), and then exactly one way to get from there through (k, r+1) to (n-r, r+1).

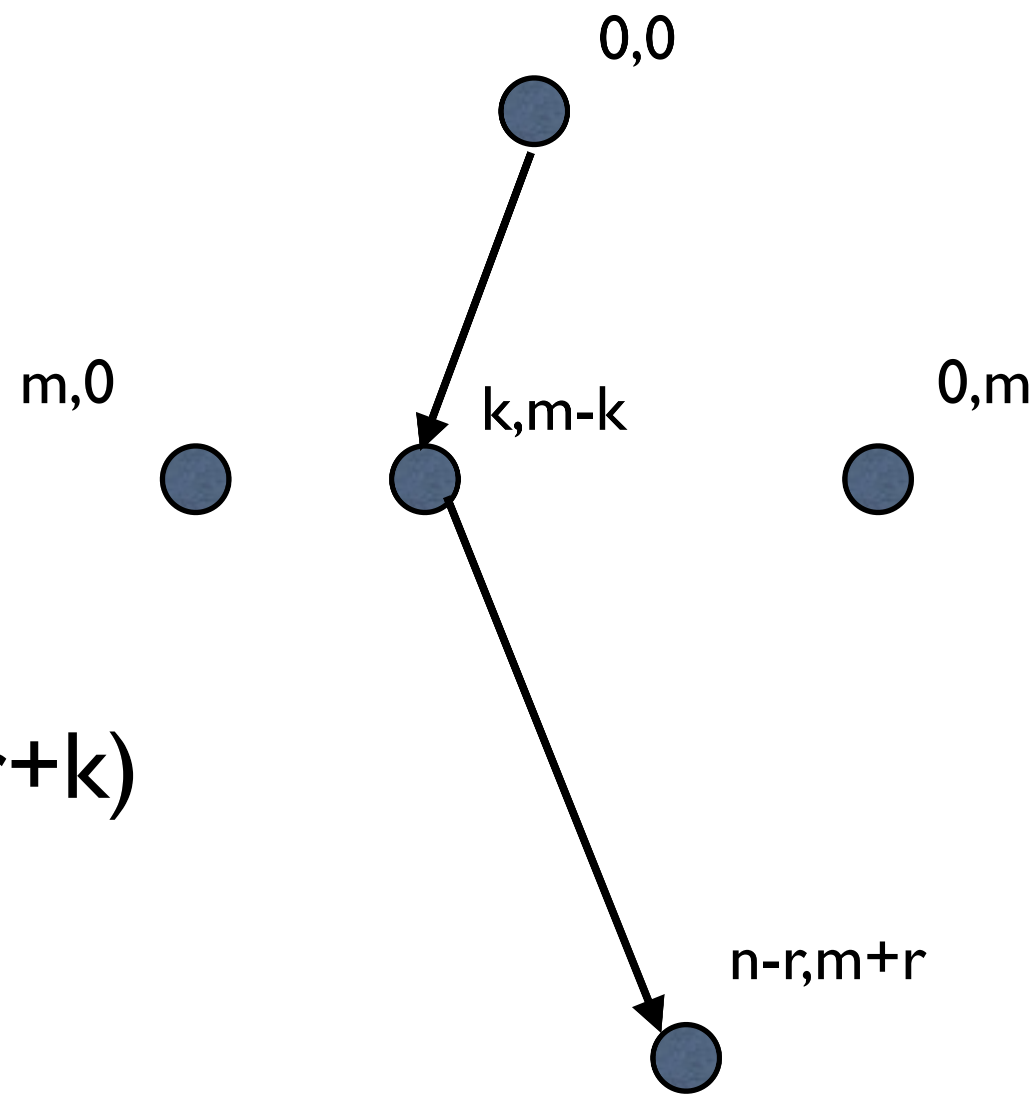
Some Proofs

- $C(n, 0)^2 + C(n, 1)^2 + \ldots + C(n, n)^2 = C(2n, n)$
- We look at all the paths from (0, 0) to (n, n). Each one must pass through exactly one of the points (0, n), (1, n-1), (2, n-2),…, (n, 0). There are C(n, k) ways to get from (0, 0) to (k, n-k), and then C(n, n-k) = C(n, k) ways to get from there to (n, n).
- By counting the same set of C(2n, n) paths in two ways, we get the identity.

One More Proof

- Sum k=0 to m of C(m, k)C(n, r+k) = C(m+n, m+r)
- C(m+n, m+r) is the number of paths from (0, 0) to (n-r, m+r). Any such path must cross the row of points from (m, 0) to (0, m) in exactly one place.
- If this point is (k, m-k), there are exactly C(m, k) ways to get there from (0, 0), and then C(n, r+k) ways to get from there to (n-r, m+r).

A Picture



Arrow moves
(n-r-k, r+k), C(n, r+k)
ways to do this

- Sum k=0 to m of C(m, k)C(n, r+k) = C(m+n, m+r)

Attacking a Sum

- Let's use binomial identities to evaluate the sum $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + (n-2)(n-1)n$.
- We can first rewrite this as $P(3, 3) + P(4, 3) + \ldots + P(n, 3)$.
- Since $P(k, 3) = 3!C(k, 3)$ for any k, our sum is also $3![C(3, 3) + C(4, 3) + \ldots + C(n, 3)]$.
- And by one of our identities, this is the same as $3!C(n+1, 4) = P(n+1, 4)/4$.
- Note the similarity to the integral of n^3 as $n^4/4$.

Attacking Another Sum

- We can use this approach and one more trick to prove a closed form for $1^2 + 2^2 + \ldots + n^2$.
- $k^2 = P(k, 2) + k$, so this sum is $[P(1, 2) + P(2, 2) + \ldots + P(n, 2)] + [1 + 2 + \ldots + n]$.
- By a similar argument to the last one, the first sum is $2!C(n+1, 3)$, and the second is $C(n+1, 2)$.
- This is $(n+1)n(n-1)/3 + (n+1)n/2 = (n+1)n[(2n-2) + 3]/6 = n(n+1)(2n+1)/6$.

Proofs by Substitution

- If we use the Binomial Theorem to compute $(1+1)^n$, we get $C(n, 0) + C(n, 1) + \ldots + C(n, n)$ because all the powers of 1 go away. This gets one of our earlier identities, for 2^n.
- Similarly, expanding $(1-1)^n$ gives us $0 = C(n, 0) - C(n, 1) + C(n, 2) - \ldots + (-1)^n C(n, n)$. This tells us that the odd-numbered C(n, i) and the even-numbered C(n, i) each add to 2^{n-1}, since they are equal to one another.

Proofs by Substitution

- For one more of these, let's look at $n(1+1)^{n-1}$.
- This is the sum for $i=0$ to $n-1$ of $nC(n-1, i)$ which is the sum of $kC(n, k)$ because $C(n, k) = (n/k)C(n-1, k-1)$ and we can take i to be $k-1$.
- Thus we get the identity $1 \cdot C(n, 1) + 2C(n, 2) + \ldots + nC(n, n) = n2^n$.