CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #16: Arrangements With Repetition
(Tucker Section 5.2, 5.3)

David Mix Barrington

19 October 2016

Arrangements w/Repetition

- Voter Power
- Arrangements and Selections
- Arranging Sets and Multisets
- Selecting Multisets
- Restrictions on Selection
- Upper and Lower Bounds
- Restricted Positions

Voter Power

- Consider a committee (or an electoral college) where different members have different numbers of votes, and decisions are made by **weighted majority**.
- You might think that voting power was proportional to the number of votes, but consider a weighting of 4, 4, 4, 4, and 1 where any three of the five members will outvote the other two.

Voter Power

- A better gauge of voter power is the **Shapley-Shubik** index, similar to the **tipping-point probability** used this season by fivethirtyeight.com.
- Look at the n! ways to order the voters, and determine which is the **median voter** in each, the one who will complete a majority if the voters are added in that order.
- The index of voter v is the fraction of orders in which v is the median voter.

Voter Power

- Clearly everyone has equal power in the 4,4,4,4,1 weighting.
- Tucker looks at 2,2,1,1,1, where there are 16 orders putting each weight-1 person in the median, and 36 for each weight-2 person.
- The six New England states are weighted 11,7,4,4,3,2 in the electoral college (if we ignore ME's split votes). Let's see the relative power of voters with these weights.

Voter Power

- MA (with 11) is the median 1/5 of the time if it is second or fifth, and all the time if it is third or fourth, for an index of 40%.
- CT (with 7) is the median 1/5 of the time if it is second or fifth, and 2/5 if is third or fourth, for an index of 20%.
- Each other state is median if it is third or fourth, with MA before it and CT after it, for an index of 10%. The four small states have equal voting power.

Arrangements and Selections

- We'll continue today with a large number of examples of counting problems, from two principle categories.
- If we have a set of objects, we can **arrange** them in some order, and ask the number of distinct ways to do this.
- We can also **select** an object, such as a set or sequence, from some category, and ask the number of ways to do this.

Arranging Sets and Multisets

- We've already seen the easy problem of selecting an order for a given set of n distinct objects. There are $P(n, n) = n!$ of them.
- We also looked last time at the number of arrangements of a given multiset with a_i copies of element x_i, and n total objects. We have $n!/a_1!a_2!\ldots a_k!$ ways to arrange these.
- It is worth looking at two different proofs that this is the right number.

Arranging a Multiset

- If we mark the a_i copies of each element x_i to distinguish them, we are left with a set of n elements, which has n! possible arrangements.
- This overcounts the arrangements of the letters themselves. If we consider two set arrangements equivalent if they resolve to the same multiset arrangement, we can easily see that there are $a_1!\dots a_k!$ set arrangements in each class.

Arrangements of Multisets

- We could also choose a multiset arrangement by first choosing one of the $C(n, a_1)$ ways to place the x_1's, then one of the $C(n-a_1, a_2)$ ways to place the x_2's, and so forth.
- Rewriting the binomial coefficients in terms of factorials gives the same answer as before.
- For example, the anagrams of "banana" can be counted as $6!/3!1!2!$, or as $C(6, 3) \times C(3, 1) \times C(2, 2) = (6!/3!3!)(3!/1!2!)(2!/2!0!) = 30$.
- The order of the choices does not matter.

Selecting Multisets

- What about *selecting* a multiset of size k from an n-element set?
- We solved this problem last time using the "stars and bars" argument. Such a multiset may be described by a string of k 0's and n-1 1's, in one of C(k+n-1, k) or C(k+n-1, n-1) ways.
- For example, there are C(15, 3) = 15×14×13 / 1×2×3 = 455 different boxes of 12 donuts taken from four different flavors.

Choosing a Multiset?

- We can choose a uniform random sequence of k objects from an n-element set by throwing an n-sided die k times. We can choose a uniform random set of k objects from an n-element set by dealing cards or drawing them out of a bag.
- There's no obvious physical way to choose a random multiset. We could choose a sequence, then ignore the order the elements came in, but this is not uniform random.

Restrictions on Selection

- We can, as Tucker does in Section 5.3, give examples of arrangement and selection problems made more complicated by restrictions.
- Suppose we have four copies of each letter a, b, c, and d, and we want to choose 10 letters out of the pool, with at least two of each.
- The only possible distributions are (4, 2, 2, 2) or (3, 3, 2, 2), or permutations thereof.

Restrictions on Selection

- With (4, 2, 2, 2), we have four choices of which letter we take four of. With (3, 3, 2, 2), we have C(4, 2) = 6 choices of which two letters to take three of, for 10 choices in all.
- That's the number of multisets. What about the arrangements? There are 10!/4!2!2!2! of each of the 4-2-2-2 multisets, and 10!/3!3!2!2! of each of the 3-3-2-2's.
- The grand total turns out to be 4×18900 + 6×25200 = 226800.

Restrictions on Selection

- Now let's say we want a dozen donuts taken from five different flavors, but with at least one donut of each flavor. How many such multisets of donuts are there?
- The easy trick here is to see that we can pick a multiset of seven donuts with no restriction, then add one of each flavor.
- So the number is $C(7+5-1,5-1) = C(11, 4) = 11\times10\times9\times8/1\times2\times3\times4 = 11\times10\times3 = 330$.

Upper and Lower Bounds

- Let's look at this same trick again by comparing two similar problems. We choose a multiset of ten balls from three colors. We first insist there are *at least* red five balls, then that there are *at most* five.
- The multisets with at least five correspond to the multisets of size five with no restrictions: there are C(5+3-1, 3-1) = 21.
- But the number with at most five is not as easy to count directly.

Upper and Lower Bounds

- We can count all multisets of ten elements, C(10+3-1, 3-1) = 66, and subtract off the C(4+3-1, 3-1) = 15 with at least six red balls to get 51 multisets with at most five.
- Or we could directly count the number with 0, 1, 2, 3, 4, and 5 red balls and add these numbers together. There are k+1 ways to make a multiset of size k with just two colors, so we have (10+1) + (9+1) + (8+1) + (7+1) + (6+1) + (5+1) = 11+10+9+8+7+6 = 51 again.

Restricted Positions

- Our last example involves the anagrams of the word "banana" with various restrictions. We counted 6!/3!1!2! = 60 of these in all.
- What if the b is followed immediately by an a? We now have an arrangement of the five-element multiset {ba, n, a, n, a}, of which there are 5!/1!2!2! = 30.
- What if the string "ban" does not occur? It's easier to count the arrangements of {ban, a, n, a} where it does, 4!/1!2!1! = 12, and subtract.

Restricted Positions

- What if the b must occur before all three a's (though not necessarily *immediately* before any of them)?
- There are a number of ways to do this. Probably the easiest is to consider the possible positions of the two n's in the string, of which there are C(6, 2) = 15.
- Once we place the n's, the other four letters must be b, a, a, and a in that order. So there is one string for each way to place the n's.