CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #15: Basic Counting Problems
(Tucker Section 5.1, 5.2)
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Basic Counting Problems

- Addition and Multiplication Rules
- Basic Examples
- Four Standard Problems
- Complications
- Arrangements With Repeated Letters
- Poker Problems
- Voter Power

Combinatorics

- We turn now to our study of counting problems, **combinatorics**.
- Our most fundamental principle is to count a set by finding a bijection between it and a set of known size.
- We analyze sets by expressing them as combinations of other sets that are easier to analyze. There are two main rules to allow this.

Addition and Multiplication

- If A and B are **disjoint** sets, meaning that $A \cap B = \emptyset$, we know that $|A \cup B| = |A| + |B|$.
- If A and B are any finite sets, we can count their **direct product**, as $|A \times B| = |A| \cdot |B|$.
- Of course these two rules are easy to prove by induction on the size of one set.

Basic Examples

- We can use these two rules to solve some basic problems.
- If we roll two six-sided dice ("roll 2D6") there are $6^2 = 36$ ways in which they may come up, from 11 to 66. Exactly four of these add to 9, for example, so the probability of throwing 9 is 4/36 = 1/9.
- To compute probabilities it is important that we find the set of events that are equally likely, here *sequences* of individual throws.

Basic Examples

- Tucker asks about choosing a pair of books from a set of 5 Spanish, 6 French, and 8 Transylvanian books. The total number of pairs is $(19 \times 18)/2 = 171$, since there are 19 ways to choose the first book, 18 ways to choose a different second book, and this procedure counts each pair exactly twice.
- To get books not in the same language, we could count $(5 \times 6) + (5 \times 8) + (6 \times 8)$ for the three ways to do it, or just subtract off the pairs in the *same* language from 171.

Counting Strings

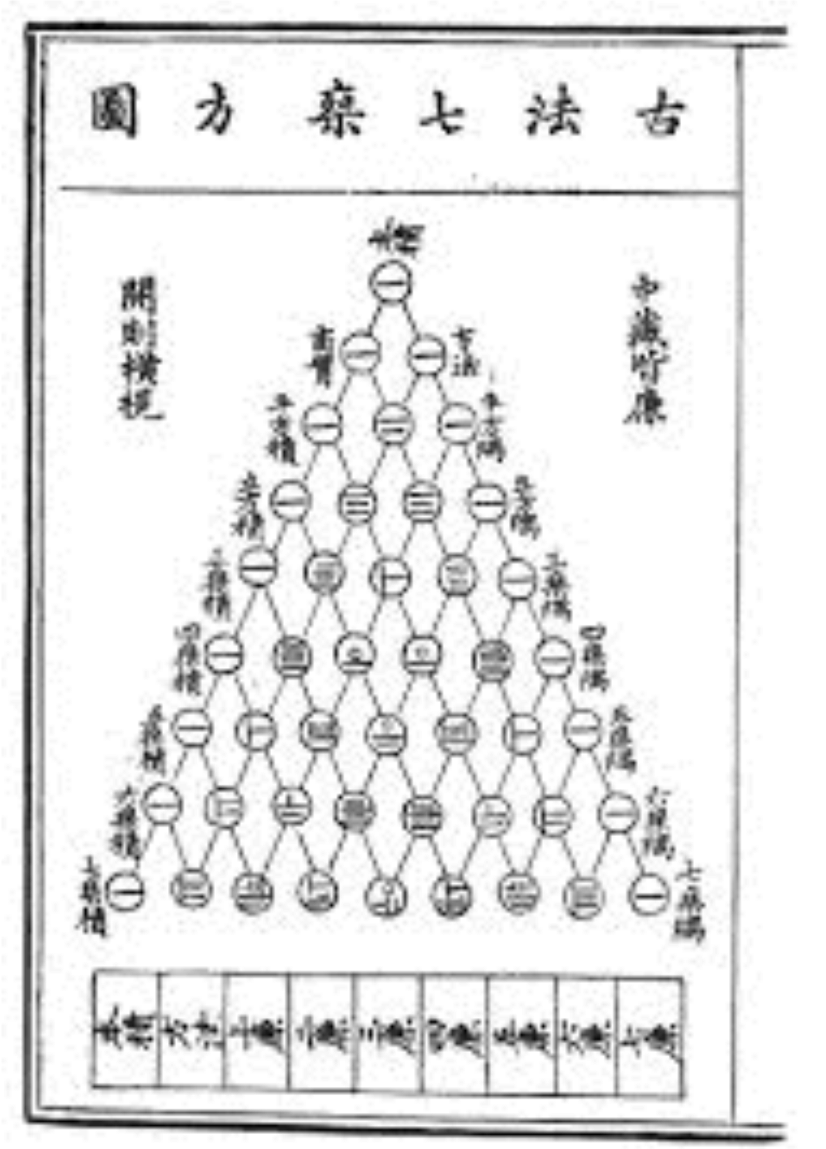
- From the product rule we can easily see that there are k^n strings of length exactly n from a k-letter alphabet. There are $k^0 + k^1 + \ldots + k^n$ strings of length *at most* n, sometimes written $k^{\leq n}$.
- Of these strings, the number with no repeated letter is k(k-1)…(k-n+1), also written $k^{\underline{n}}$ and called "k to the n falling".
- These latter strings have n! representatives of each of the *sets* of n letters in the alphabet.

Four Counting Problems

- Choosing k items out of a set of n, we can care about order or not, allow repeats or not.
- First problem: Order counts, repeats, direct use of Product Rule gives n^k.
- Second Problem: Order counts, no repeats, successive choices give $P(n, k) = n!/(n-k)!$, also called "n to the k falling" or $n^{\underline{k}}$.
- Third Problem: No order, no repeats, counting subsets, correcting for overcount gives $C(n, k)$.

Binomial Coefficients

- C(n, k) is P(n, k)/k! or n!/k!(n-k)!.
- Easiest calculation for C(6, 3), for example, is $6 \cdot 5 \cdot 4 / 1 \cdot 2 \cdot 3 = 20$.
- Pascal's (or Yang Hui) Triangle holds the value of C(n, k) as entry k in row n.
- Various identities can be observed from the triangle, and we will prove them later.



Counting Multisets

- Fourth Problem: No order, repeats allowed.
- We are counting **multisets** of size k taken from a set of size n.
- The "stars and bars" argument forms a bijection between such a multiset and a string of k stars and n-1 bars. The third problem's solution tells us there are C(n+k-1, k) of these.

Stars and Bars

- Let's look at n = 5 and k = 3. A multiset can be described by a sequence of numbers in {1,2,3,4,5} in sorted order. 225, for example, represents two copies of 2 and one of 5.
- To get a binary string from 225, we take a 0 for each entry and put a 1 between each pair of values. So 225 becomes 1001110.
- To go the other way, we convert each 0 in the binary string to a number based on the number of 1's occurring before it.

Complications

- We can make these problems more complicated by insisting that a particular letter come in a particular position, or that it occur a certain number of times in the multiset, or that certain letters are adjacent.
- In each case we match the new problem to one of standard type. For example, if we want strings from {a,b,c,d,e} containing at least one ab substring, we look at abxxx, xabxx, xxabx, and xxxab, correcting for any double counting.

Repeated Letters

- Tucker asks about counting the **anagrams** of the word SYSTEMS, meaning permutations of the multiset.
- If the three S's were different, there would be 7!, so the correct number is 7!/3! = 840 when we correct for overcounting.
- How many have the three S's together? Just the 5! permutations of {SSS, Y, T, E, M}.

Poker Problems

- With dice it is sequences that are equally likely, while with cards it is sets.
- A **poker hand** is a subset of the 52 cards with exactly five elements. There are (52 choose 5) = 2598560 of these.
- A **full house** is a set with three cards of one rank and two of another. (There are 13 ranks with four cards each.)

Poker Problems

- To count full houses, we pick the rank with three (13), then the rank with two (12), then which three ((4 choose 3) = 4), then which two ((4 choose 2) = 6, for 13×12×4×6 = 3744.
- To count **two-pair** hands we choose which ranks have pairs ((13 choose 2) = 78), which pair of each rank (6^2 = 36), which rank for the odd card (11) and which odd card (4), for 78×36×11×4 = 123552.
- Look carefully at the double-counting!

Voter Power

- Consider a committee (or an electoral college) where different members have different numbers of votes, and decisions are made by **weighted majority**.
- You might think that voting power was proportional to the number of votes, but consider a weighting of 4, 4, 4, 4, and 1 where any three of the five members will outvote the other two.

Voter Power

- A better gauge of voter power is the **Shapley-Shubik** index, similar to the **tipping-point probability** used this season by fivethirtyeight.com.
- Look at the n! ways to order the voters, and determine which is the **median voter** in each, the one who will complete a majority if the voters are added in that order.
- The index of voter v is the fraction of orders in which v is the median voter.

Voter Power

- Clearly everyone has equal power in the 4,4,4,4,1 weighting.
- Tucker looks at 2,2,1,1,1, where there are 16 orders putting each weight-1 person in the median, and 36 for each weight-2 person.
- The six New England states are weighted 11,7,4,4,3,2 in the electoral college (if we ignore ME's split votes). Let's see the relative power of voters with these weights.

Voter Power

- MA (with 11) is the median 1/5 of the time if it is second or fifth, and all the time if it is third or fourth, for an index of 40%.
- CT (with 7) is the median 1/5 of the time if it is second or fifth, and 2/5 if is third or fourth, for an index of 20%.
- Each other state is median if it is third or fourth, with MA before it and CT after it, for an index of 10%. The four small states have equal voting power.