CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #12: Minimum Spanning Trees
(Tucker Section 4.2)
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Minimum Spanning Trees

- Definitions and Motivation
- The Prim and Kruskal Algorithms
- An Example
- Correctness of the Algorithms
- Implementation of the Algorithms
- Using an MST to Approximate TSP
- A Better Approximation

Definitions and Motivation

- Suppose we have a weighted undirected graph where nodes are towns, edges are roads, and weights are the cost of plowing a road.
- We have a limited budget and can only plow some of the roads, but we need to make it possible to get from any town to any other.
- We want a subset of the edges forming a tree containing all the nodes, a **spanning tree**.

Minimum Spanning Trees

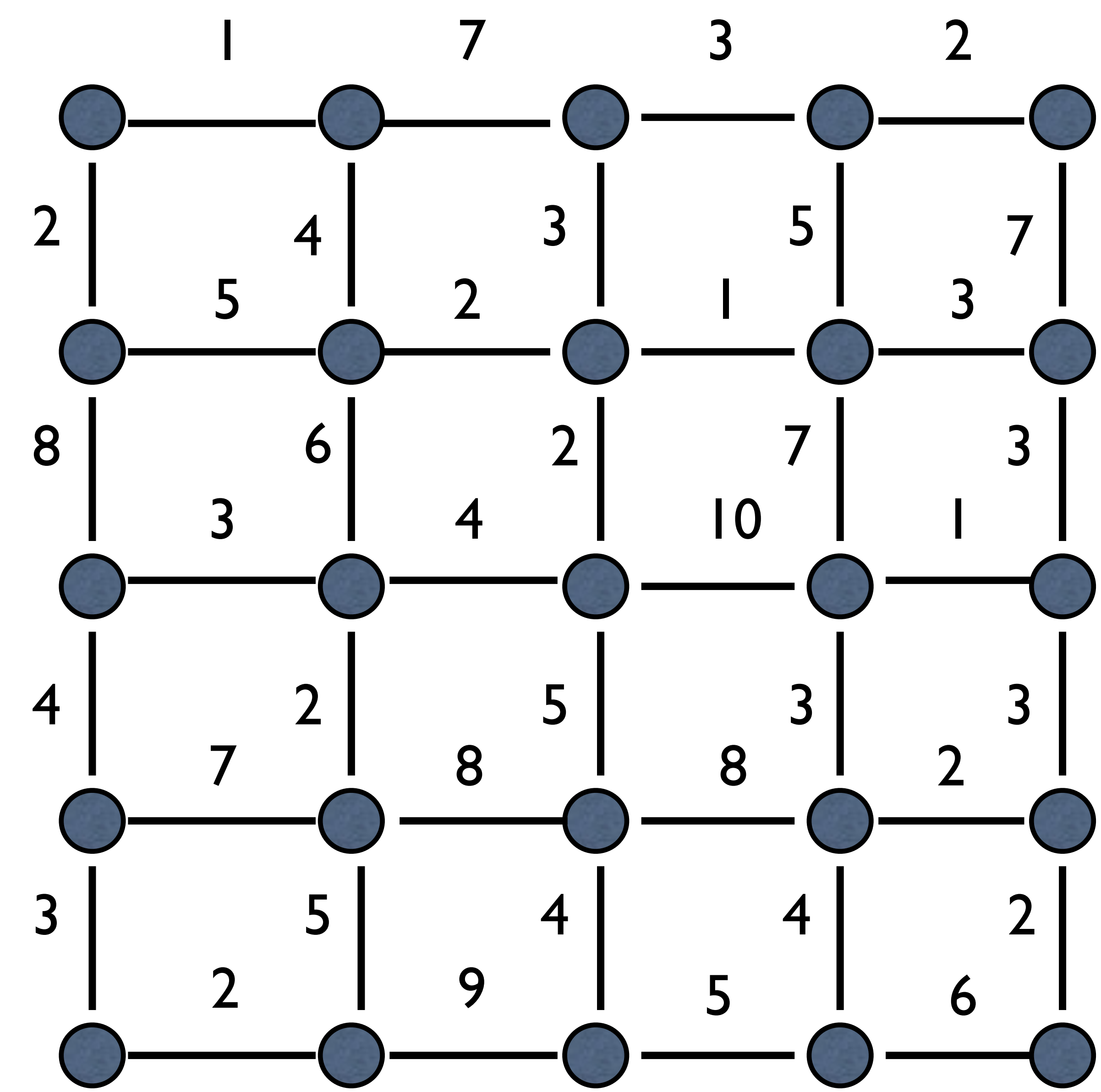
- A spanning tree has a weight, the sum of the weights of its edges. A minimum spanning tree is one such that no other has less weight.
- There may be more than one MST for a given graph. In fact, if each edge has weight 1, then every spanning tree is an MST.
- We'll assume for the rest of the lecture that the original graph is connected and that all the weights are positive.

Two Algorithms for MST

- We'll present two algorithms to find an MST. Both are **greedy** algorithms, considering all the edges of a certain type and taking the edge of minimum weight.
- **Prim's Algorithm** builds a single tree by starting with one node, then repeatedly adding the cheapest edge that connects a node in the tree to one outside of it.
- **Kruskal's Algorithm** always takes the cheapest edge that does not form a cycle with the existing edges.

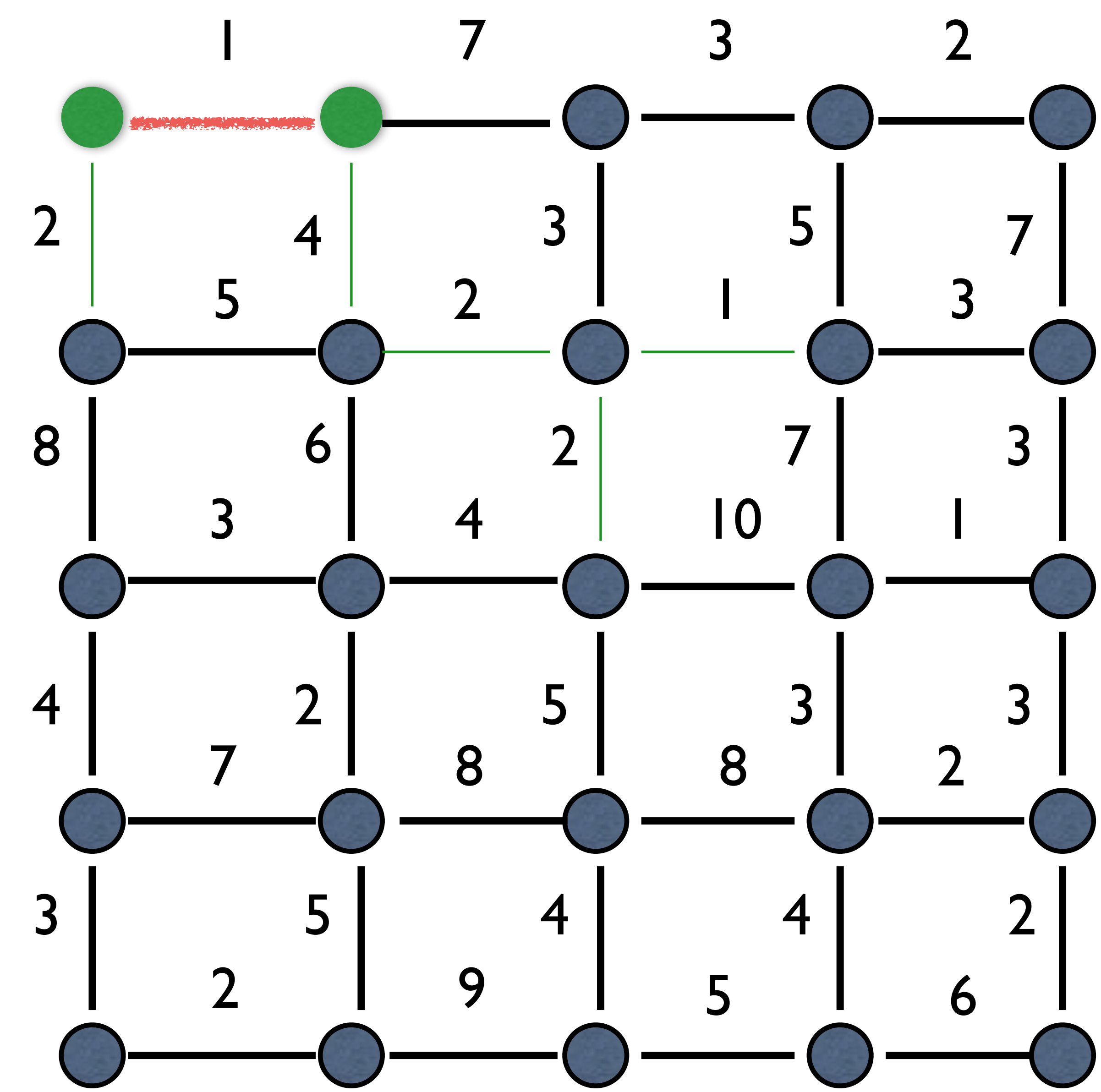
An Example

- Here's a weighted graph with 25 nodes and 40 edges.
- We'll run both algorithms in turn to get a minimum spanning tree for it.
- We'll start Prim with the first 1-edge.



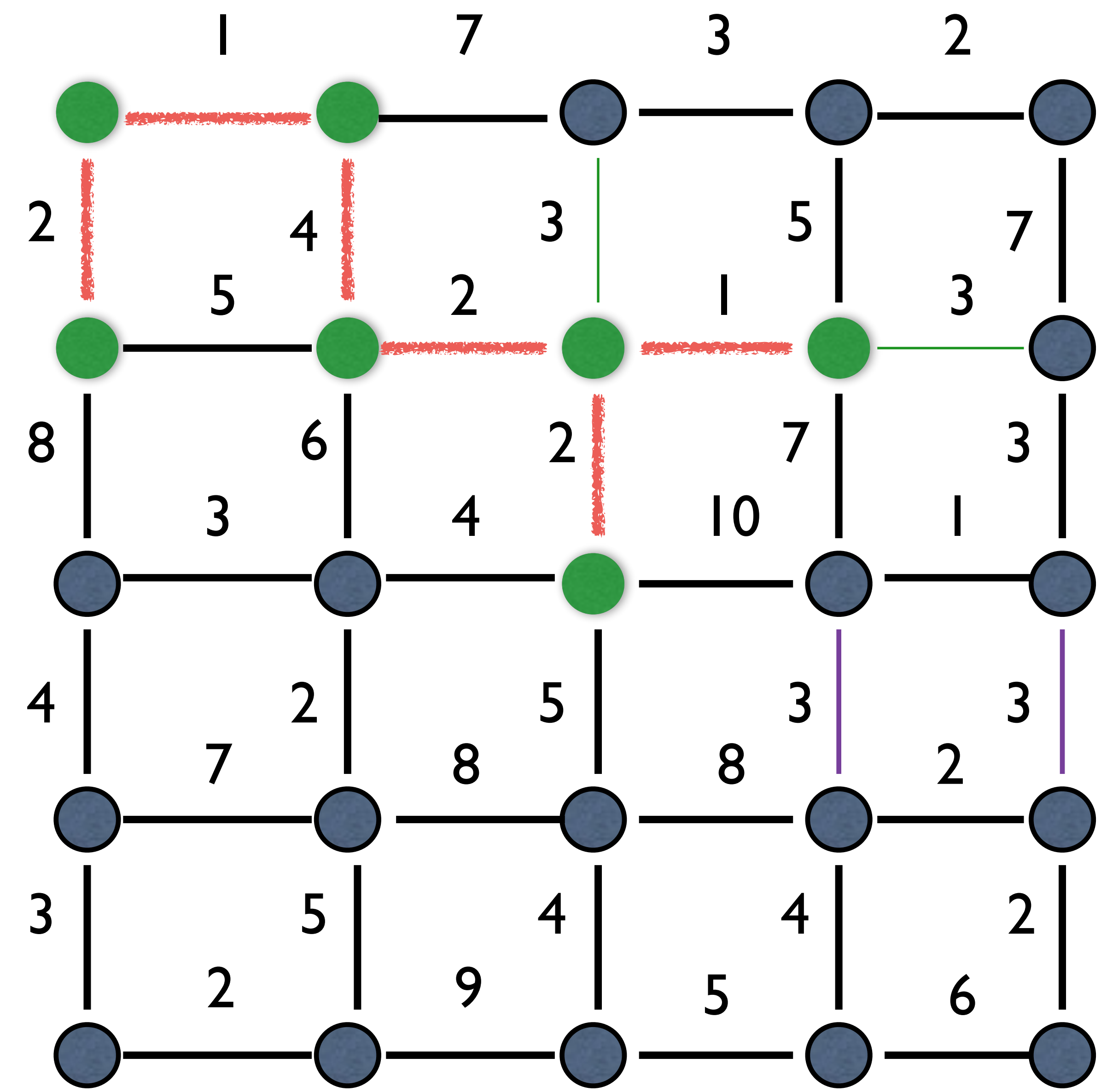
An Example

- The nodes in green and edges in red are in the tree.
- Prim goes on to take edges with one green endpoint, cheapest first. The next five it takes, in order 2, 4, 2, 1, 2, are shown in green.



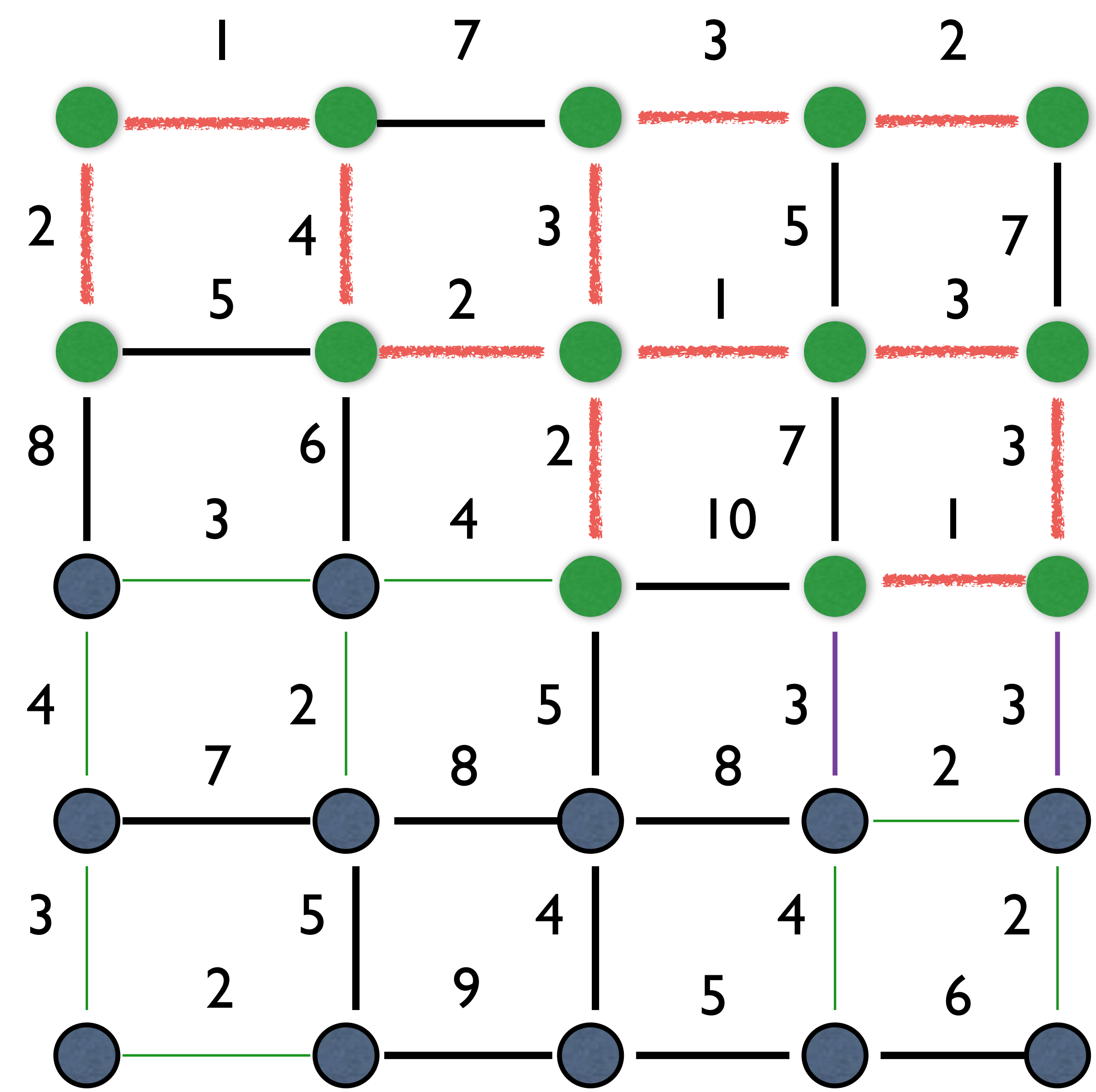
An Example

- At this point we have a choice of size-3 edges.
- That choice doesn't affect the tree, but a later one could.
- It turns out that either of the purple edges could be in the MST.



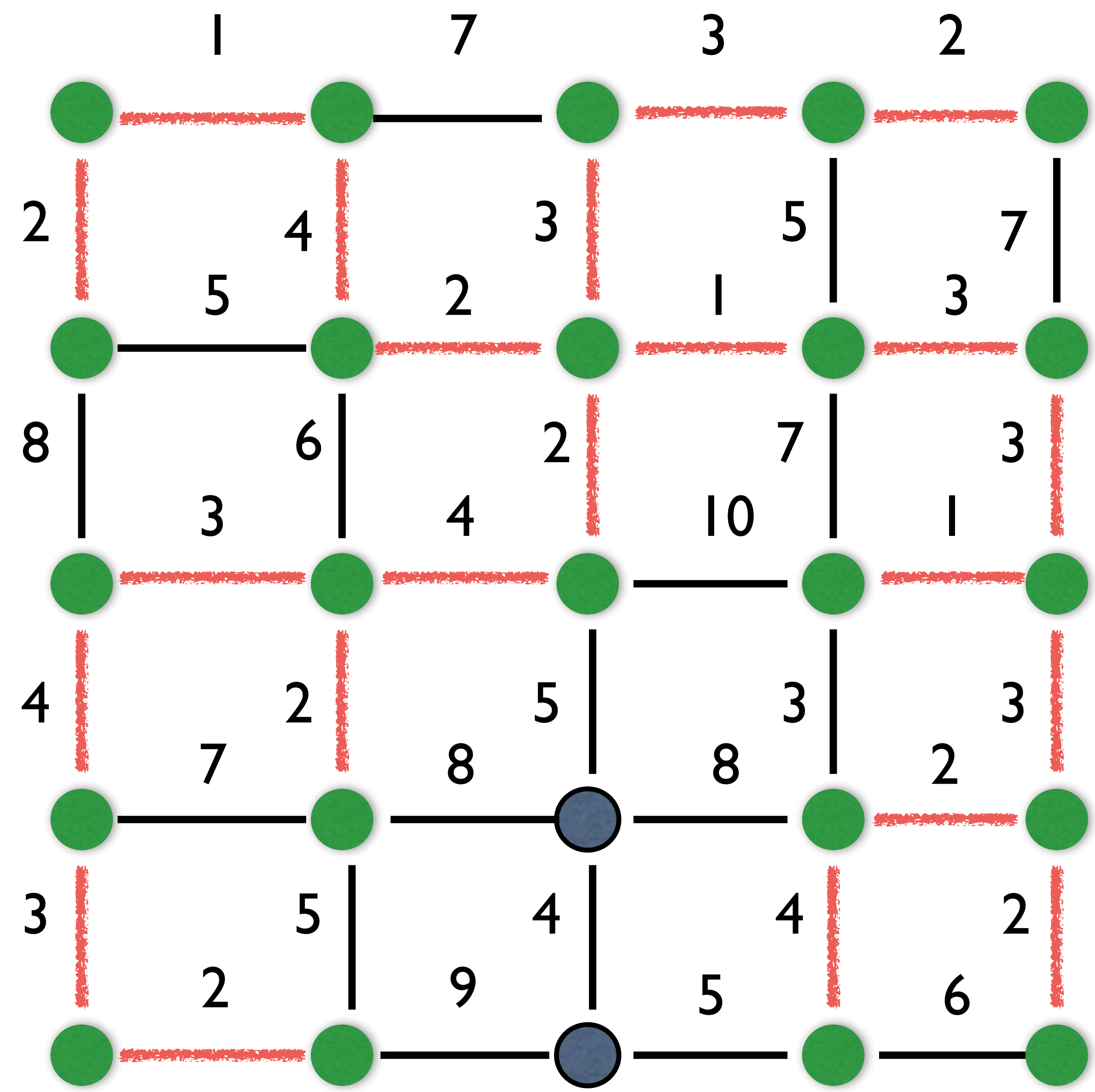
An Example

- It turns out that either of the purple edges could be in the MST.
- Once we make this choice, we'll get the next edges in green.



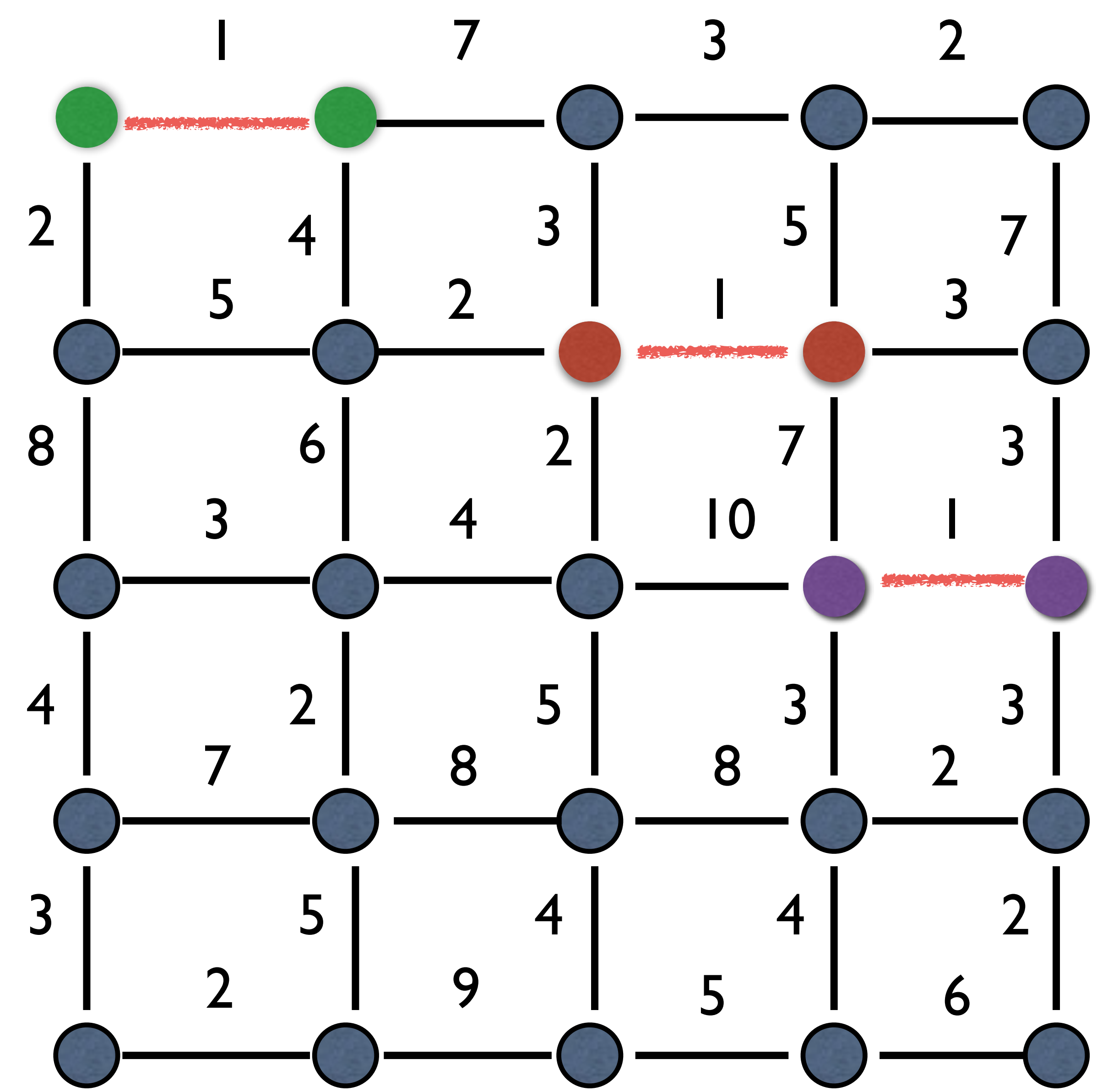
An Example

- Now there are two ways to get the last two nodes, both with the same cost.
- We've used 24 edges of total weight 65.



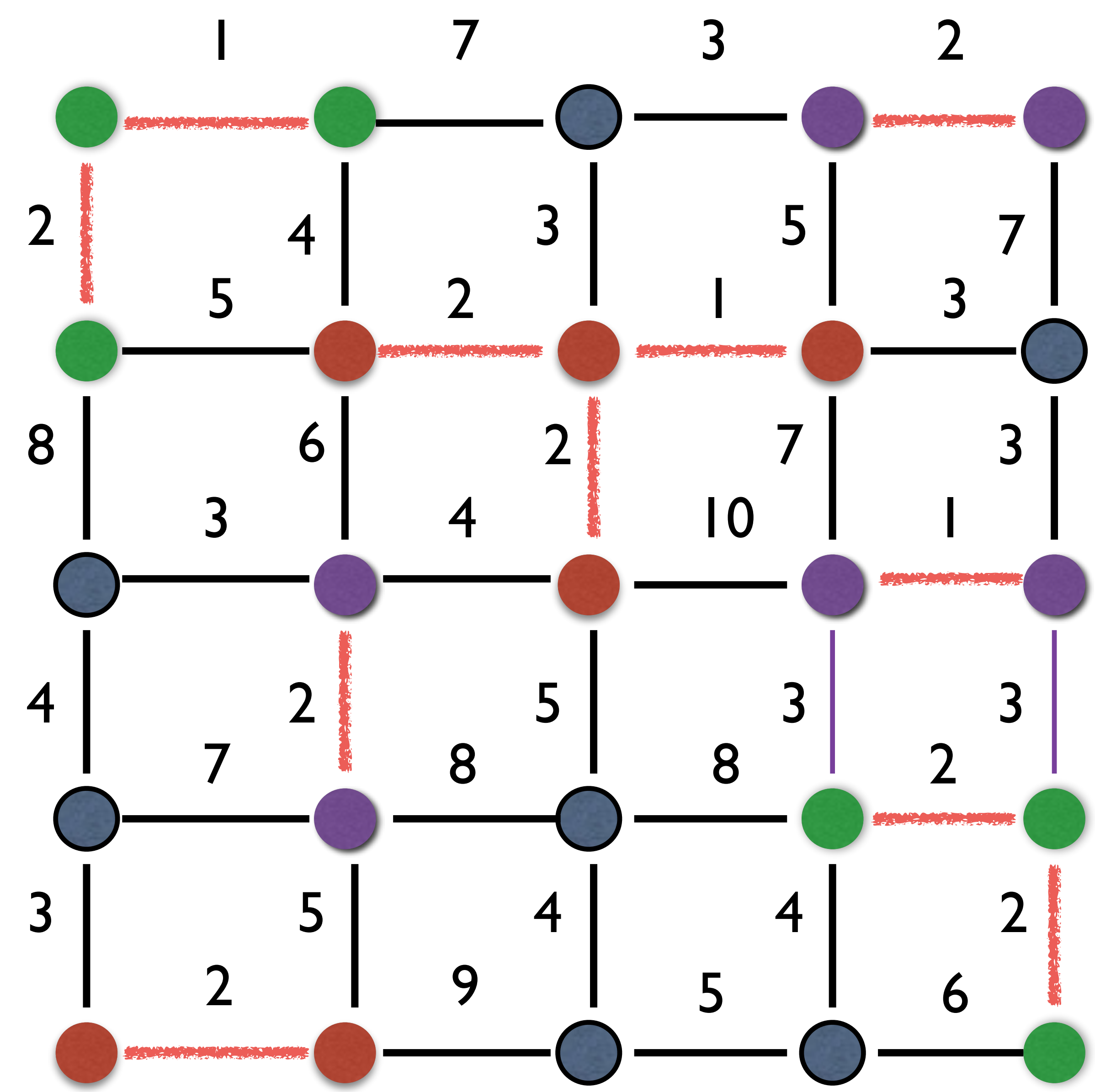
An Example

- To begin Kruskal, we take all the edges of weight 1 because we don't form a cycle.
- We start building up connected components.
- The weight-2 edges also don't form any cycle.



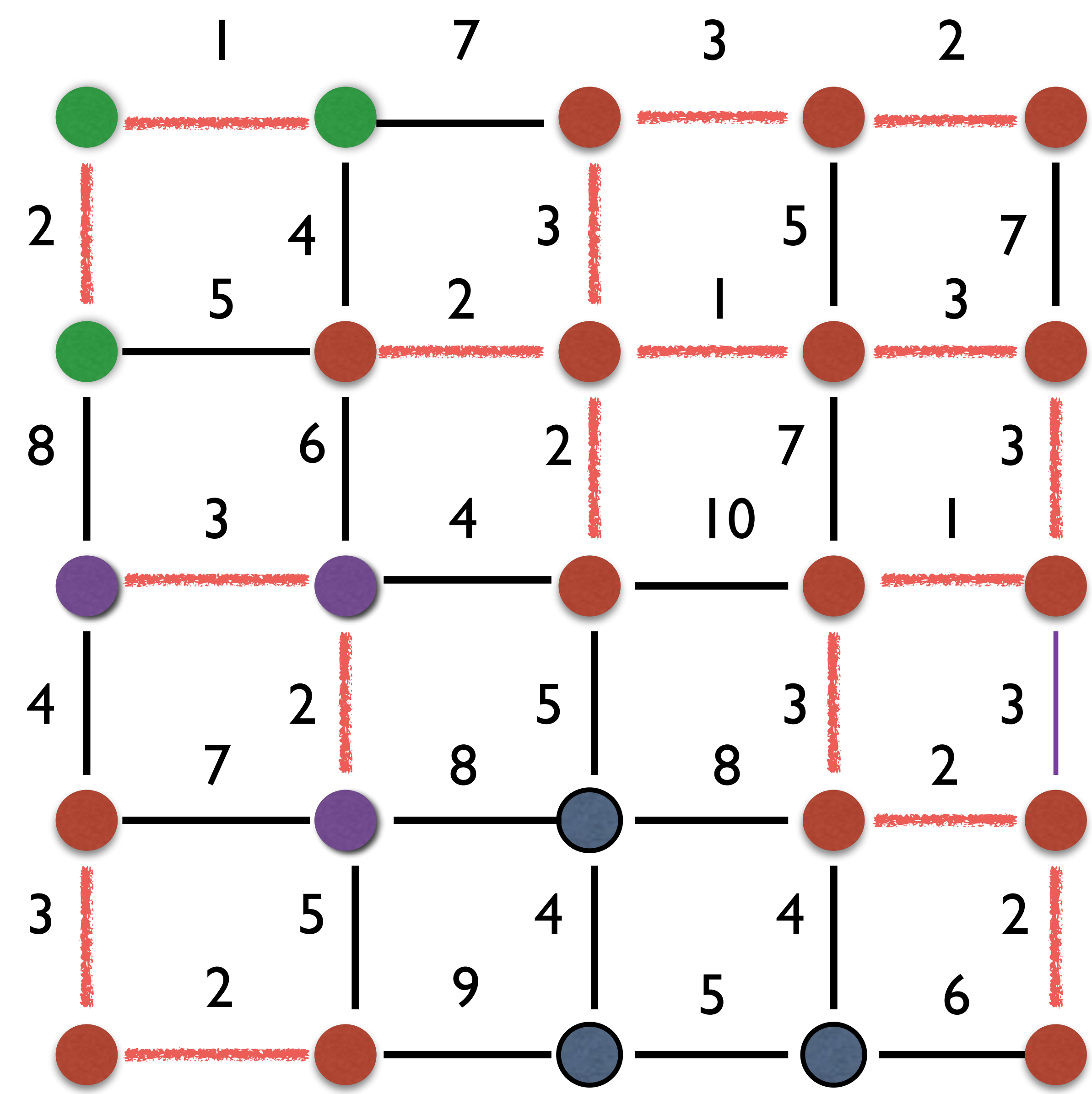
An Example

- The weight-2 edges also don't form any cycle. (We've now added them in the diagram.)
- But the 3-edges will. We can't add both the vertical 3-edges in purple, but we can add either one.



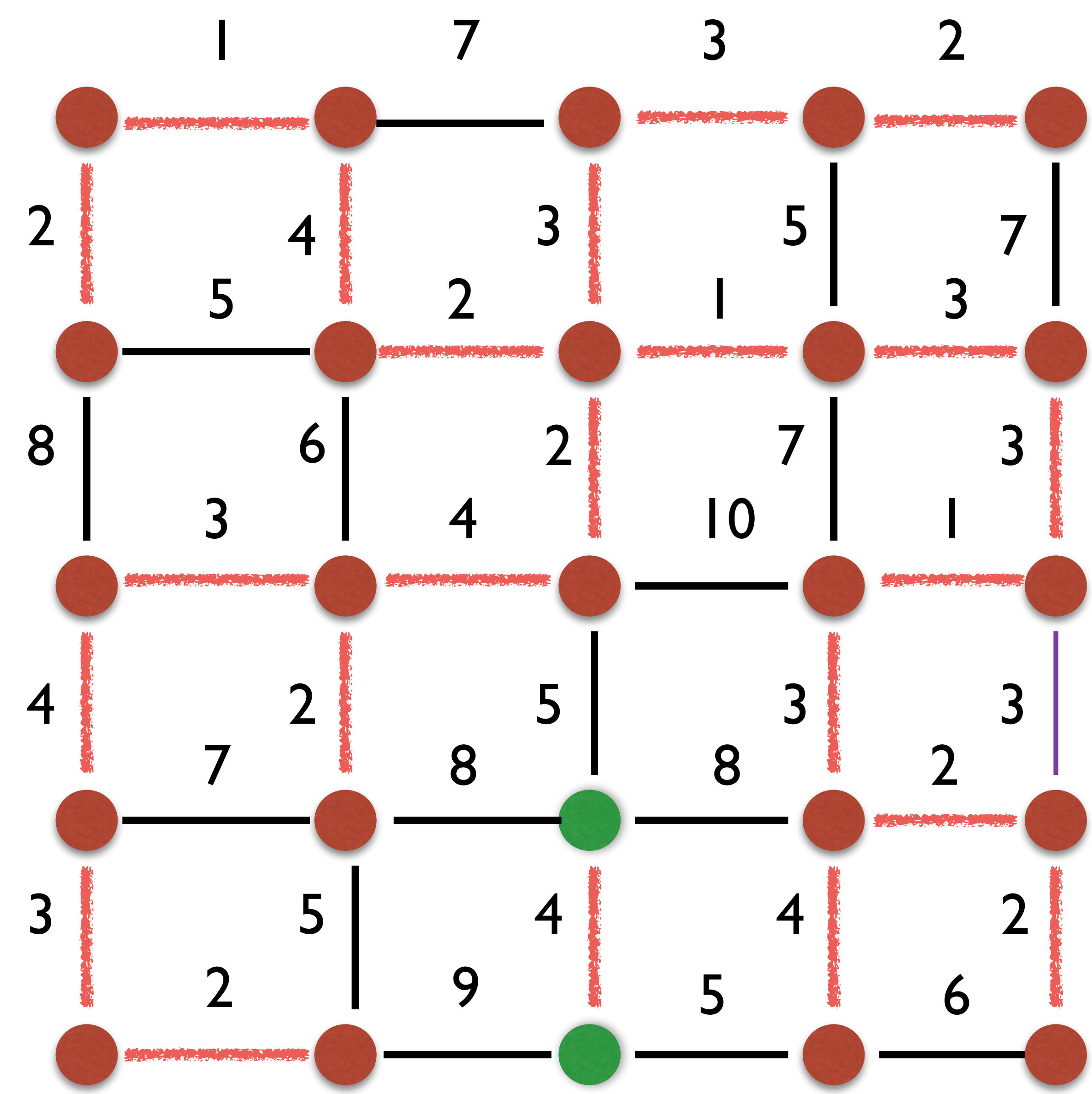
An Example

- Here we did not take the purple 3-edge because we took the 3-edge to its left first.
- Now we have only 7 connected components left. The five 4-edges don't form cycles, and reduce us to only two.



An Example

- We'll complete the spanning tree with one of the two 5-edges with one red and one green endpoint.
- There were four possible trees we might have produced, each of total weight 65.



Correctness of Prim

- Assume (for simplicity) that all the edge weights are distinct.
- We'll get a contradiction from the assumption that some tree T' has smaller weight than the tree T^* produced by Prim.
- Let e = (u, v) be the first edge not in T' that we put into T^*, and consider the point at which we did so. Let X be the nodes in T^* at that point. We know that e is cheaper than any other edge with one endpoint in X.

Correctness of Prim

- Since T′ is a spanning tree, there is a path from u to v in it, and this path must leave X by some edge e′.
- Since e′ has one endpoint in X and one out of it, it must have weight larger than that of e.
- Now replacing e′ with e gives a spanning tree with weight less than that of T′, a contradiction. (On HW#3 you will show that this replacement always yields a tree.)

Correctness of Kruskal

- We can similarly show that Kruskal is correct. Again consider an MST T′. As Kruskal adds edges to its edge set, e = (u, v) be the first edge it adds that is not in T′, and let Z be the edges in the set at that point (before e is added).

- Node u and v are in different connected components of the forest Z. T′ must have a path from u to v, and this path must contain an edge e′ that connects two components of Z.

- Once again replacing e′ with e in T′ gives a spanning tree smaller than the alleged MST T′.

Implementation and Time

- How do we implement these algorithms?
- For Prim we can use a priority queue like that in UCS. We put edges into the PQ when they are found. At each stage we start pulling out the minimum-weight edge in the PQ, rejecting them it if both endpoints are in X, until we find exactly the edge we need.if not it is the edge needed by Prim.
- Overall we have O(e) PQ operations for O(e log e) total time.

Implementation and Time

- With Prim we could tell whether to reject an edge using a flag on each vertex to say whether it was in X.
- But with Kruskal we want to reject an edge if it forms a cycle with existing edges, which happens if the endpoints are in the *same connected component* of the forest made by the existing edges.

Implementation and Time

- This is the **dynamic transitive closure** problem, to maintain the set of connected components as new edges are introduced.
- We don't want to, say, DFS the graph again for each new edge.
- In CS 311 you will probably see the **union-find algorithm**, which solves this problem in nearly linear time. With that, the running time of Kruskal is also about O(e log e).

Applying MST to TSP

- Recall that Tucker presented an algorithm to approximate the least-weight Hamilton circuit in a weighted graph, getting a tour whose weight was at most twice the optimal weight. The weights were symmetric and obeyed the triangle inequality.
- In such a graph, we can always **shortcut** two edges (u, v) and (v, w) with a single edge (u, w), without adding weight.

Applying MST to TSP

- Consider an MST of this weighted graph. Change each edge into two directed edges, and make an Euler tour E of the resulting directed graph.
- The weight of E is twice that of the MST. And since dropping any edge from the optimal tour C^* gives a spanning tree, the weight of E is at most twice that of C^*.
- Shortcutting E leads to a Hamilton tour with no more weight than that of E.

A Better Approximation

- The nodes O of odd degree in the MST must have a perfect matching. (Why?) Later we will see how to find the minimum-weight matching in polynomial time.
- If we add the matching edges to the MST, we get a graph where all vertices have even degree, and there must be an Euler tour. Shortcutting this tour gets us a Hamilton tour.

A Better Approximation

- We already know that the weight of the MST is less than that of the optimal tour C^*.
- The nodes of O divide C^* into an even number of paths. If we two-color these, one color set has weight less than half that of C^*. And the shortcutting of this is a matching with no more weight.
- So the total weight of our eventual Hamilton tour is at most 3/2 that of C^*.