NAME: ______________________

SPIRE ID: ______________________

COMPSCI 250
Introduction to Computation
Second Midterm Spring 2025

D. A. M. Barrington and M. Golin 14 April 2025

DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-12, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 13 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page – both sides are scanned.
- But, if you do write on the back, you must explicitly add a note on the front side stating that you are continuing on the back page. Otherwise, we might not see your solution on Gradescope.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.
- Your answers must be LEGIBLE, and not cramped. Write only short paragraphs with space between paragraphs

1	/20+5
2	/20
3	/40
4	/20
Total	/100+5

Question 1:

(A) (10 points)

According to legend, a king in India was so pleased with the new game of chess that he offered the inventor anything he asked. The inventor asked for a few grains of rice to be placed on a chessboard for him to take home. More specifically, he asked for one grain of rice to be placed on the first square of the board, two on the second, four on the third, and so on. Thus, 2^{i-1} grains would be placed on the i'th square.

Let $S(n)$ be the total number of grains of rice that the inventor receives after rice is placed on n squares.

Prove, by ordinary induction on all positive naturals, that $S(n) = 2^n - 1$.

Please note that $2^n - 1$ and 2^{n-1} are in general different numbers. Also, your proof MUST be by induction. You may not use the formula for geometric series.

i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal, explicitly identify where the induction hypothesis is being usesd and justify every one of your manipulation steps.

If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

(B) (10 points) Recall the Fibonacci numbers F_n defined on the naturals n by: $F_0 = 0$, $F_1 = 1$, and,

$$\forall n \geq 1,\ F_{n+1} = F_n + F_{n-1}. \tag{1}$$

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7
0	1	1	2	3	5	8	13

Let $S_n = \sum_{i=0}^{n} F_i$ be the sum of the first $n+1$ Fibonacci numbers:
Example: $S_5 = 0+1+1+2+3+5 = 12$. Notice that $S_5 = F_7 - 1$.

Prove, *using induction*, that, for all positive naturals n, $S_n = F_{n+2} - 1$.

i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal, explicitly identify where the induction hypothesis is being usesd and justify every one of your manipulation steps.

If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

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(C) Extra Credit (5 points) For natural n, define A_n by $A_0 = 2$, $A_1 = 1$ and

$$\forall n \geq 1,\ A_{n+1} = A_n + 6A_{n-1}. \tag{2}$$

As examples, $A_2 = 1 + 6 \cdot 2 = 13$ and $A_3 = 13 + 6 \cdot 1 = 19$.

Prove, *using induction*, that $A_n = 3^n + (-2)^n$.

As a sanity check, note that $A_3 = 19 = 27 - 8 = 3^3 + (-2)^3$.

i) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(ii) Next, write your base case(s) in the box below.

(iii) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations will have points deducted.
Be sure to clearly describe your induction goal, explicitly identify where the induction hypothesis is being usesd and justify every one of your manipulation steps.

If you run out of space, continue the proof on the back page (with a note stating that you are writing on the back).

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Question 2 (20): Induction 2

A **rooted ternary tree (RTT)** is constructed as follows:

R0: It is either a single node, which is its root, or

R1: It is a new node, its root, which is connected to the roots of exactly three other RTT's.

Furthermore,

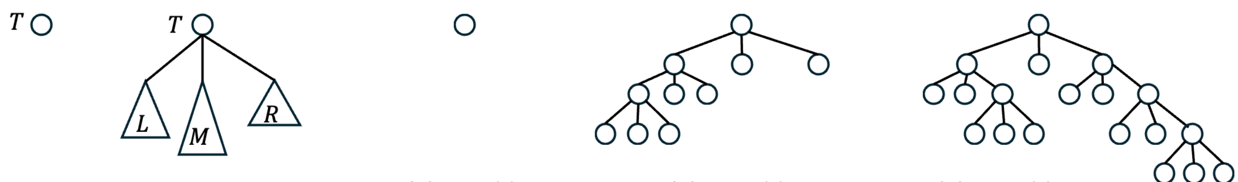
R2: The only RTT's are those made by the first two rules.

Let $\mathcal{T}$ represent the set of all RTT's. For tree $T \in \mathcal{T}$,

$\lvert T \rvert$	$=$	the number of nodes in T,	$L(T)$	$=$	the number of leaves in T,
$I(T)$	$=$	the number of internal nodes in T,	$d(T)$	$=$	the depth of T.

Recall that the depth of T is the length of the longest path from the root to any leaf.

Diagrams (i-ii) are illustrations of the rules. Diagrams (iii), (iv) and (v) are examples of some RTT's and their associated values.



(i) (ii) (iii) $\lvert T \rvert = 1, L(T) = 1$, $I(T) = 0, d(T) = 0$ (iv) $\lvert T \rvert = 10, L(T) = 7$, $I(T) = 3, d(T) = 3$ (v) $\lvert T \rvert = 19, L(T) = 13$, $I(T) = 6, d(T) = 4$

Parts (A) and (B) of this problem are on the following pages. When writing the solutions to parts (A) and (B), you must use the the mathematical notation we provided above.

If your proof has multiple pieces, place each piece in a separate paragraph with space between the paragraphs. Be sure to clearly describe your induction goal.

If you run out of space, continue the proof on the back page of the problem (with a note stating that you are writing on the back).

Family Name: ____________________

(A, 10 points) Prove, by induction, that for every $T \in \mathcal{T}$, it is true that $L(T)$ is odd.

a) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(b) Next, write your base case(s) in the box below.

(c) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

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(B, 10 points) Prove, by induction, that for every $T \in \mathcal{T}$, it is true that $|T| \leq \frac{3^{d(T)+1}-1}{2}$.

a) First write your induction hypothesis in the box below. This should be in the form $P(x)$, where you *must* explicitly explain what x is and write an unambiguous statement of $P(x)$.

(b) Next, write your base case(s) in the box below.

(c) Finally, provide your induction step. This step will be marked on how clear and mathematically precise your proof is. Ambiguous explanations or explanations missing details will have points deducted.

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Question 3 (40): In this problem, we are going to carry out four searches on three graphs: an undirected graph U, a directed graph D, and a labeled directed graph G. All represent the same nine nodes, and each has the same twelve edges, but they have the edge types appropriate to their type of graph.

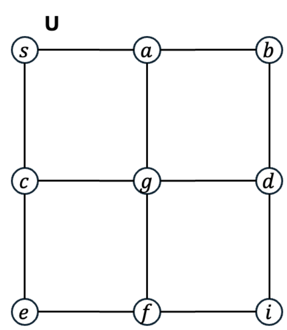
- (A, 10) **Undirected Breadth-First Search:**

  (i) Carry out a **BFS search** for the **undirected** graph U, starting with node s and with **no goal node**.

  List all the events that occur in the running of the BFS, in the order in which they occur. An event is either the placement of a list-item on the open list or the removal of a list-item off the open list.

  Describe list-items in the form of pairs such as "(a, b)", meaning that the node a went on the list while node b was processed.

  When two or more list-items need to come off the open list, and they entered at the same time, take the one first that comes earlier alphabetically.



(ii) In addition to writing the list, **also draw** the BFS tree from this search, starting with node s and with no goal node. Indicate the tree edges and the non-tree edges.

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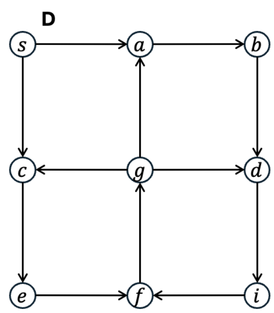
(B, 10) **Directed Depth-First Search:**

(i) Carry out a **DFS search** for the **directed** graph D, starting with node s and **with no goal node**.

List all the events that occur in the running of the DFS, in the order in which they occur. An event is either the placement of a list-item on the open list or the removal of a list-item off the open list.

Describe list-items in the form of pairs such as "(a, b)", meaning that the node a went on the list while node b was processed.

When two or more nodes need to come off the open list, and they entered at the same time, take the one first that comes earlier alphabetically.



(ii) In addition to writing the list, **also draw** the DFS tree from this search, starting with node s and with no goal node. For each non-tree edge, **identify** it as a back, cross, or forward edge.

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(C, 10) **Uniform-Cost Search:**

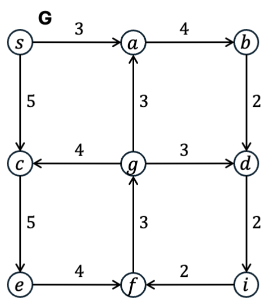
Conduct a uniform-cost search of the labeled directed graph G, with s as the start node and g as the goal node.

List all the events that occur in the running of the UCS, in the order in which they occur.

An event is either the placement of a list-item on the open list or the removal of a list-item off the open list.

Describe list-items using triples such as "(a, t, b)", meaning that this node a went on the list while b was processed, with t being its priority value.

When two or more list-items need to come off the list, they should follow the rules of uniform-cost, and you may break ties as you like.



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(D, 10) A^* **Search:**

We will now ask you to conduct an A^* search for the directed graph G, with s as the start node and g as the goal node, using the following heuristic function h. For any node x, $h(x)$ will be the distance from x to g *ignoring the direction of the edges.* We could have you calculate that by a UCS of the undirected version of the graph, but we will just give you the answers:

$$\begin{array}{lllll} h(s) = 6, & h(a) = 3, & h(b) = 5, & h(c) = 4, & h(g) = 0 \\ h(d) = 3, & h(e) = 7, & h(f) = 3, & h(i) = 5 & \end{array}$$

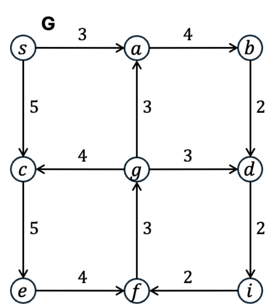
Conduct an A^* search of the directed graph D to determine the shortest distance from s to g, using the heuristic function h above, with s as the start node and g as the goal node.

List all the events that occur in the running of the A^* search, in the order in which they occur.

An event is either the placement of a list-item on the open list or the removal of a list-item off the open list.

Describe list-items using tuples such as "(a, t, d, b)", meaning that this node a went on the list while b was processed, with t being its priority value and d being the distance from s to a.

When two or more list-items need to come off the list, they should follow the rules of uniform-cost, and you may break ties as you like.



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Question 4 (20) The following are ten true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing.

After reading the questions, write the correct answer, either T (for true) or F (for false), in the corresponding column.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)

- (a) Let $A \subseteq \{a, b\}^*$ be a language defined by the two rules $\lambda \in A$ and $\forall w : (w \in A) \rightarrow [(awa \in A) \wedge (bwb \in A)]$, such that the only strings in A are those given by those rules. Then there does not exist any string in A with odd length.

- (b) Let $P(n)$ be a predicate on the naturals. If $P(0)$ is true, and $\forall n : P(n) \rightarrow (P(3n) \wedge P(3n+1) \wedge P(3n+2))$ is true, then $P(n)$ is true for all naturals n.

- (c) Let T be a boolean expression tree, where the only operators used are $\neg$ and $\oplus$. If we change the value of one of its leaves from true to false, or vice versa, then the value of T also changes from true to false, or vice versa.

- (d) Let u and v be two distinct nodes in a directed graph G such that there exists a directed path from u to v. If all directed edges have the same cost, then we can use BFS starting at u to find the path from u to v with the smallest cost.

- (e) If $h(x)$ is an admissible heuristic function for an A^* search of an undirected labeled graph, then $h(x)$ must be at least as large as the shortest-path distance from x to the goal node.

- (f) Let T be any tree, and consider making a rooted tree from it by choosing one of its nodes as the root. Then if v has strictly more neighbors than any other node, choosing v as the root gives a rooted tree with the smallest possible depth.

- (g) Consider the set Σ^* of all strings over the alphabet $\Sigma = \{a\}$. Then the concatenation operations for such strings is both commutative and associative.

- (h) Consider undirected graphs with exactly four nodes. Any such graph with exactly four edges is connected, but there exist such graphs with four nodes and three edges that are not connected.

- (i) Consider a game tree (as defined in Lecture 27),where there is exactly one leaf marked "B" for Black victory, all the other leaves are marked "W" for White victory, the root is not a leaf, every internal node has exactly two children, and White moves first. Then White has a winning strategy for this game.

- (j) Consider the boolean expression with infix representation $(a \wedge b) \vee (c \wedge d)$. Then the prefix and postfix representations of this expression are reversals of one another.

COMPSCI 250 Second Midterm Supplementary Handout: 14 April 2025

Definitions from Question 2:

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R1: It is a new node, its root, which is connected to the roots of exactly three other RTT's.

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R2: The only RTT's are those made by the first two rules.

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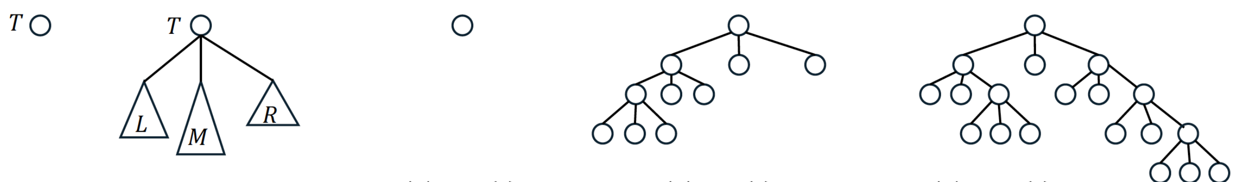

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The Graphs from Question 3.