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COMPSCI 250
Introduction to Computation
SOLUTIONS to First Midterm Spring 2022

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-7, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 8 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/30
3	/30
4	/25+5
Total	/100+5

Questions 1 and 2 involve a set $D = \{b, r\}$ of dogs, consisting of Blaze and Rhondda, and a set $T = \{A, D, M, N, O\}$ of toys, consisting of an Antler, a Dinosaur, a Marrowbone, a Nylabone^{TM}, and an Octopus. The predicate $H(x, Y)$ means "dog x currently has the toy Y". Note that it is possible for both dogs to have the same toy at the same time, or even more than one at the same time.

Question 1 (15): Translate each statement as according to the directions:

- (a, 2) (to English) (Statement I) $H(r, D) \rightarrow (H(r, O) \wedge \neg H(r, N))$
  If Rhondda has the Dinosaur, then she has the Octopus but not the Nylabone.
- (b, 2) (to symbols) (Statement II) If Rhondda has the Octopus, then she has the Dinosaur, and if she does not have the Dinosaur, then she does not have the Octopus.
  $(H(r, O) \rightarrow \neg H(r, D)) \wedge (H(r, D) \rightarrow \neg H(r, O))$, *which can be simplified to either implication alone, or to* $\neg H(r, D) \vee \neg H(r, O)$.
- (c, 2) (to English) (Statement III) $\neg(H(r, D) \wedge H(r, N)) \rightarrow H(r, O)$
  If it is not the case that Rhondda has both the Dinosaur and the Nylabone, then she has the Octopus.
- (d, 3) (to symbols) (Statement IV) There is exactly one toy in T that Blaze does not have.
  $\exists Y : \forall Z : \neg H(b, Y) \wedge (\neg H(b, Z) \rightarrow (Z = Y))$
- (e, 3) (to English) (Statement V) $H(r, O) \rightarrow \exists Y : H(r, Y) \wedge (Y \neq O)$
  If Rhondda has the Octopus, then she also has another different toy.
- (f, 3) (to symbols) (Statement VI) Some toy is being held by both dogs.
  $\exists Y : H(b, Y) \wedge H(r, Y)$ *or* $\exists Y : \forall d : H(d, Y)$.

Question 2 (30): These questions use the definitions, predicates, and premises on the supplementary sheet.

- (a, 10) Assuming *only* that Statements I, II and III are true, determine the truth values of the three propositions $p = H(r, D)$, $q = H(r, N)$, and $r = H(r, O)$. You may use a truth table or a deductive sequence proof. Make sure that there is exactly one solution.

  The three statements translate to $p \rightarrow (\neg q \wedge r)$, $r \rightarrow p$, *and* $\neg(p \wedge r) \rightarrow r$.

  Case 1: p *is false. By Statement II,* r *is false, which by Statement III,* p *and* q *are both true, and because of* p *we have a contradiction.*

  Case 2: p *is true. Statement I tells us that* q *is false and* r *is true.*

  So $p \wedge \neg q \wedge r$ *is the only possible combination, and this satisfies Statement I trivially, Statement II trivially, and Statement III vacuously.*

- (b, 20) Now assume **only that Statements III, IV, V** are assumed true. Again, **do not assume Statements I, II, or VI** for this problem.

  Prove that Statement VI is true. You may use either English or symbols, but make it clear each time you use a quantifier proof rule.

  Case 1: Rhondda does not have the Octopus. By Statement III, Rhondda has both the Dinosaur and the Nylabone. We Instantiate Y to be the toy that Blaze does not have. If $Y = D$, we can Specify Z to N and see that Blaze has the Nylabone and so both dogs have it, and we can prove Statement VI by using Existence on N. If $Y \neq D$, we Specify Z to D and see that Blaze has the Dinosaur, and similarly we can prove Statement VI by Existence on D.

  Case 2: Rhondda has the Octopus. By Statement V, there exists another toy X that she also has. We again Instantiate Y to the toy that Blaze does not have. As before, if $Y = O$, then both dogs have X, and otherwise both dogs have O. So either way we prove Statement VI by Existence on the toy that both dogs have.

Question 3 (30): The following are fifteen true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them use the sets, relations, and functions defined on the supplemental sheet, but you should assume the truth of Statements I-VII only if explicitly told to.

- (a) A binary relation R can be symmetric and transitive but not reflexive.
  TRUE, the empty relation is a counterexample
- (b) If $\neg p \wedge q$ is true, then the statement $\neg(p \vee \neg q)$ is also true.
  TRUE, DeMorgan Rule
- (c) If we know that $\neg x \rightarrow \neg y$ is false, then $y \rightarrow x$ must be true.
  FALSE, it must be false
- (d) If the following statements are true, "All Clownfish like to swim" and "All fish who are not Clownfish like to swim", then the statement "All fish like to swim" must also be true.
  TRUE, Proof By Cases
- (e) If we have two congruences, $i \equiv j \pmod{x}$ and $i \equiv j \pmod{y}$, we can use the Chinese Remainder Theorem to say that $i \equiv j \pmod{xy}$.
  FALSE, though the converse is true
- (f) Say that we have a function E from set X to set Y, and E is one-to-one. This implies E is also onto.
  FALSE, we have lots of examples with one but not the other
- (g) In all meaningful ways the `int` type in java is equivalent to the set of integers.
  FALSE, Java ints have wraparound
- (h) If you can prove $D(x)$ with x being arbitrary, then the Rule of Generalization allows us to derive $\forall x : D(x)$.
  TRUE, that's just what it says.
- (i) It is possible for a Hasse diagram with seven nodes to have no edges at all.
  TRUE. The partial order would be the identity relation.
- (j) Given the sets $A = \{7, \{2\}, \{\{13, 2\}\}\}$, $B = \{13, 2, 7\}$, and $C = \{2\}$, we see that $B \subseteq A$ and $C \in A$.
  FALSE. The first claim is wrong – only one of the three elements of B are in A. The second claim is true.
- (k) The symmetric difference of two sets C and D is the set of all elements that is neither in C or D.
  FALSE. The symmetric difference are the elements in C or in D, but not in both.
- (l) Say we have three finite sets X, Y and Z, where $|X| < |Y|$ and $|Y| > |Z|$. Let us define a function from X to Z as the composition of one function from X to Y and another function from Y to Z. Then we know this function from X to Z can not be both one-to-one and onto.
  FALSE. Any set with $|X| = 1$, $|Y| = 2$, and $|Z| = 1$ will be a counterexample.
- (m) The Least Common Multiple of two relatively prime numbers x and y will always be $x \times y$.
  TRUE. We always have $gcd(x, y) \times lcm(x, y) = x \times y$.

- (n) The set of natural numbers is closed under the operation of subtraction.

  FALSE. "Closure" would mean that for any two naturals x and y, $x - y$ would be a natural. But, for example, $2 - 1$ is not a natural.

- (o) Let A be any non-empty set and let P be any unary predicate over A. Then the statement $(\forall x : P(x)) \rightarrow (\exists x : P(x))$ is true.

  TRUE. Choose any element a of A, as we can since A is non-empty. Specify the premise to get $P(a)$, then use Existence.

Question 4 (25+5): Here are a variety of number theory questions.

- (a, 5) Of the two numbers 26 and 28, one has a multiplicative inverse modulo 63 and the other does not. Which is which? Prove your answer.

  The EA finds $63\%26 = 11$, $26\%11 = 4$, $11\%4 = 3$, $4\%3 = 1$, $3\%1 = 0$, *so there is an inverse. The EA finds* $63\%28 = 7$, $28\%7 = 0$, *so* 28 *has no inverse. We could also note directly that* 7 *is a common factor.*

- (b, 10) Find the inverse in part (a) that exists.

  $63 = 2 \times 26 + 11$, $26 = 2 \times 11 = 4$, $11 = 2 \times 4 = 3$, $4 = 1 \times 3 = 1$, *so the linear combinations are* $63 = 1 \times 63 + 0 \times 26$, $26 = 0 \times 63 + 1 \times 26$, $11 = 1 \times 63 - 2 \times 26$, $4 = -2 \times 63 + 5 \times 26$, $3 = 5 \times 63 - 12 \times 26$, $1 = -7 \times 63 + 17 \times 26$, *The inverse of* 26, *modulo* 63, *is* 17.

- For the remainder of the problem, let m and n be any two positive naturals and let P be the set of all pairs (x, y) where x and y are naturals with $x < m$ and $y < n$. If k is any natural, the function f_k is a function from naturals in the set $\{0, 1, \ldots, k-1\}$ to P, defined by the rule $f(i) = (i\%m, i\%n)$.

- (c, 5) Whether the function f_k is a bijection depends on k, m, and n. If there is any value of k that makes it a bijection for a particular m and n, what must the value of k be? Justify your answer, using facts we have used about bijections.

  If the function is a bijection, the sizes of the domains and codomains must be the same. Since P has size mn, and the domain has size k, we must have $k = mn$.

- (d, 5) For what values of m and n can we get a bijection? Prove your answer.

  We get a bijection if and only if m and n are relatively prime. If they are, the Chinese Remainder Theorem tells us that each pair (x, y) in P is formed by the remainders of $z\%m$ and $z\%n$ if and only z is congruent to some c modulo mn, with c depending on x and y. If they are not relatively prime, and so $g > 1$ is a common divisor of m and n, there can not be any natural i such that $i\%m = 0$ and $i\%n = 1$, since if i if a multiple of m, it is divisible by g, but if $i\%n = 1$, it is not divisible by g.

- (e, 5XC) Suppose that m and n are such that there is no bijection from any f_k to P. What is the largest value of k such that f_k is an injection? Justify your answer.

  Let g be the greatest common divisor of m and n. The correct value of k is mn/g. The first value of k where the injection fails is when $f(k) = f(j)$ for some $j < k$, and this will occur when $f(k) = f(0)$ (since if $f(k) = f(j)$ were true for $j > 0$, $f(k-j) = f(0)$ would be true, and k would not be the first counterexample. This $f(k)$ would be the least common multiple of m and n, which we saw is mn/g.