NAME: ____________________________

# COMPSCI 250
# Introduction to Computation
# First Midterm Spring 2022

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are four problems on pages 2-7, some with multiple parts, for 100 total points plus 5 extra credit. Final scale will be determined after the exam.
- Page 8 contains useful definitions and is given to you separately – do not put answers on it!
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

| | |
|---|---|
| 1 | /15 |
| 2 | /30 |
| 3 | /30 |
| 4 | /25+5 |
| Total | /100+5 |

Questions 1 and 2 involve a set $D = \{b, r\}$ of dogs, consisting of Blaze and Rhondda, and a set $T = \{A, D, M, N, O\}$ of toys, consisting of an Antler, a Dinosaur, a Marrowbone, a Nylabone$^{TM}$, and an Octopus. The predicate $H(x, Y)$ means "dog $x$ currently has the toy $Y$". Note that it is possible for both dogs to have the same toy at the same time, or even more than one at the same time.

**Question 1 (15):** Translate each statement as according to the directions:

- (a, 2) (to English) (Statement I) $H(r, D) \rightarrow (H(r, O) \wedge \neg H(r, N))$
- (b, 2) (to symbols) (Statement II) If Rhondda has the Octopus, then she has the Dinosaur, and if she does not have the Dinosaur, then she does not have the Octopus.
- (c, 2) (to English) (Statement III) $\neg(H(r, D) \wedge H(r, N)) \rightarrow H(r, O)$
- (d, 3) (to symbols) (Statement IV) There is exactly one toy in $T$ that Blaze does not have.
- (e, 3) (to English) (Statement V) $H(r, O) \rightarrow \exists Y : H(r, Y) \wedge (Y \neq O)$
- (f, 3) (to symbols) (Statement VI) Some toy is being held by both dogs.

**Question 2 (30):** These questions use the definitions, predicates, and premises on the supplementary sheet.

- (a, 10) Assuming *only* that Statements I, II and III are true, determine the truth values of the three propositions $p = H(r, D)$, $q = H(r, N)$, and $r = H(r, O)$. You may use a truth table or a deductive sequence proof. Make sure that there is exactly one solution.

- (b, 20) Now assume **only that Statements III, IV, V** are assumed true. Again, **do not assume Statements I, II, or VI** for this problem.

  Prove that Statement VI is true. You may use either English or symbols, but make it clear each time you use a quantifier proof rule.

**Question 3 (30):** The following are fifteen true/false questions, with no explanation needed or wanted, no partial credit for wrong answers, and no penalty for guessing. Some of them use the sets, relations, and functions defined on the supplemental sheet, but you should assume the truth of Statements I-VII only if explicitly told to.

- (a) A binary relation $R$ can be symmetric and transitive but not reflexive.
- (b) If $\neg p \wedge q$ is true, then the statement $\neg(p \vee \neg q)$ is also true.
- (c) If we know that $\neg x \rightarrow \neg y$ is false, then $y \rightarrow x$ must be true.
- (d) If the following statements are true, "All Clownfish like to swim" and "All fish who are not Clownfish like to swim", then the statement "All fish like to swim" must also be true.
- (e) If we have two congruences, $i \equiv j \pmod{x}$ and $i \equiv j \pmod{y}$, we can use the Chinese Remainder Theorem to say that $i \equiv j \pmod{xy}$.
- (f) Say that we have a function $E$ from set $X$ to set $Y$, and $E$ is one-to-one. This implies $E$ is also onto.
- (g) In all meaningful ways the `int` type in java is equivalent to the set of integers.
- (h) If you can prove $D(x)$ with x being arbitrary, then the Rule of Generalization allows us to derive $\forall x : D(x)$.
- (i) It is possible for a Hasse diagram with seven nodes to have no edges at all.
- (j) Given the sets $A = \{7, \{2\}, \{\{13, 2\}\}\}$, $B = \{13, 2, 7\}$, and $C = \{2\}$, we see that $B \subseteq A$ and $C \in A$.
- (k) The symmetric difference of two sets $C$ and $D$ is the set of all elements that is neither in $C$ or $D$.
- (l) Say we have three finite sets $X$, $Y$ and $Z$, where $|X| < |Y|$ and $|Y| > |Z|$. Let us define a function from $X$ to $Z$ as the composition of one function from $X$ to $Y$ and another function from $Y$ to $Z$. Then we know this function from $X$ to $Z$ can not be both one-to-one and onto.
- (m) The Least Common Multiple of two relatively prime numbers $x$ and $y$ will always be $x \times y$.
- (n) The set of natural numbers is closed under the operation of subtraction.
- (o) Let $A$ be any non-empty set and let $P$ be any unary predicate over $A$. Then the statement $(\forall x : P(x)) \rightarrow (\exists x : P(x))$ is true.

**Question 4 (25+5):** Here are a variety of number theory questions.

- (a, 5) Of the two numbers 26 and 28, one has a multiplicative inverse modulo 63 and the other does not. Which is which? Prove your answer.

- (b, 10) Find the inverse in part (a) that exists.

- For the remainder of the problem, let $m$ and $n$ be any two positive naturals and let $P$ be the set of all pairs $(x, y)$ where $x$ and $y$ are naturals with $x < m$ and $y < n$. If $k$ is any natural, the function $f_k$ is a function from naturals in the set $\{0, 1, \ldots, k-1\}$ to $P$, defined by the rule $f(i) = (i\%m, i\%n)$.
- (c, 5) Whether the function $f_k$ is a bijection depends on $k$, $m$, and $n$. If there is any value of $k$ that makes it a bijection for a particular $m$ and $n$, what must the value of $k$ be? Justify your answer, using facts we have used about bijections.
- (d, 5) For what values of $m$ and $n$ can we get a bijection? Prove your answer.
- (e, 5XC) Suppose that $m$ and $n$ are such that there is no bijection from any $f_k$ to $P$. What is the largest value of $k$ such that $f_k$ is an injection? Justify your answer.

**COMPSCI 250 First Midterm Supplementary Handout: 12 October 2021**

Here are definitions of sets, predicates, and statements used on the exam.

Remember that the scope of any quantifier is always to the end of the statement it is in.

In Question 4 we often use the Java operator % on naturals, so that $x\%y$ is the remainder when $x$ is divided by $y$.

Remember that an **injection** is a function that is one-to-one, and that a **bijection** is a function that is both onto and one-to-one.

Questions 1 and 2 involve a set $D = \{b, r\}$ of dogs, consisting of Blaze and Rhondda, and a set $T = \{A, D, M, N, O\}$ of toys, consisting of an Antler, a Dinosaur, a Marrowbone, a Nylabone$^{TM}$, and an Octopus. The predicate $H(x, Y)$ means "dog $x$ currently has the toy $Y$". Note that it is possible for both dogs to have the same toy at the same time, or even more than one at the same time.

We have abbreviated the four propositions $p = H(r, D)$, $q = H(r, N)$, and $r = H(r, O)$.

The six statements of Question 1 are:

- (a, 2) (to English) (Statement I) $H(r, D) \rightarrow (H(r, O) \wedge \neg H(r, N))$
- (b, 2) (to symbols) (Statement II) If Rhondda has the Octopus, then she has the Dinosaur, and if she does not have the Dinosaur, then she does not have the Octopus.
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