CMPSCI 250: Introduction to Computation

Lecture #2: Propositions and Boolean Operators
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Propositions and Operators

- What is a Proposition?
- Java Boolean Variables
- Boolean Operators, Compound Propositions
- AND, OR, NOT, and XOR
- Implication and Equivalence
- Tautologies

Formal Languages

- Any set of strings over Σ is called a **language** over Σ. We can define languages using any of our ways of defining sets.
- Let Σ = {0, 1}. Define the language X to be all strings that have a 1 at the beginning, a second different 1 at the end, and 0's in the middle. We can write X as {11, 101, 1001, 10001,...} or as {w: w starts and ends with 1 and has no other 1's}. Later we'll call this language "10^*1".

Decision Problems

- The **decision problem** for a language X is to input a string w (over Σ, the correct alphabet) and return a boolean that is true if $w \in X$ and false if not.
- Given a language, how difficult is it for a computer to solve its decision problem? This is the central question of **formal language theory**. We'll touch on this at the end of the course.

What is a Proposition?

- A **proposition** is a statement that is either true or false.
- In mathematics we want to reason about statements like "x = 5" or "these two triangles are congruent" without knowing whether they are true or false. We could say "if x = 5, then $x^2 = 25$", or "if one length and all three angles are the same, then the triangles are congruent".

More about propositions

- In computing we reason with **assertions** about a program, like "if this method terminates, the value of `i` is positive". Ultimately we'd like to say "if the input is as specified, then the output is as specified", meaning "the program is **correct**".
- What isn't a proposition? Questions, commands, statements without meaning, paradoxes like "this statement is false", or incompletely specified statements.

Java Boolean Values

- Java has a primitive `boolean` data type, and every boolean has either the value `true` or the value `false`.
- We use booleans in the conditions for `if` or `while` statements -- if we write "`if (x > 4) y = 5;`", then the statement "`y = 5`" will be executed only if the boolean value "`x > 4`" evaluates to `true` at run time.

Java Boolean Operators

- The operators ==, `!=`, >, >=, <, and <= create boolean values from values of other types. We often write methods that return `boolean` values, or use existing `boolean` methods like `equals`. We'll soon see operators that make new booleans from old.
- You may think of a "proposition" as any statement that could be modeled by a boolean variable. Of course, propositions may be about anything, not just computer data.

Making Compound Propositions

- A **compound proposition** is a proposition that is made up from other propositions, called **atomic propositions**, using **boolean operators**.
- If I say "you must have MATH 132, and either CMPSCI 187 or ECE 242", we can define three atomic propositions and write this as a compound proposition. We let x represent "you have MATH 132", y be "you have CMPSCI 187, and z be "you have ECE 242".

Making Compound Propositions

- Now my statement can be written "x, and either y or z". Symbolically, we write this as "$x \wedge (y \vee z)$".
- If x, y, and z are any three booleans, the truth of $x \wedge (y \vee z)$ depends on which of x, y, and z are true. In Java, if x, y, and z are boolean variables, we can write the expression `x && (y || z)`, and this represents $x \wedge (y \vee z)$.
- This is the **propositional calculus**.

AND and OR

- If x and y are any two propositions, their **conjunction** x $\wedge$ y is the proposition that is true if and only if *both* x and y are true. We read it "x and y". The Java operators & and && both compute the value of a conjunction -- we usually use && which only evaluates the second argument if it is needed.
- The **disjunction** of x and y is written x $\vee$ y, read "x or y", and is true if either is true, or both. In Java the disjunction is | or ||.

Practice Clicker Question #1

- Let p be "dogs like beef", q be "cats like tuna", and r be "pigs like mud". Which of the following English statements matches "$(r \vee q) \wedge (p \vee q)$"?
- (a) Either pigs like mud or cats like tuna, and either dogs like beef or cats like tuna.
- (b) If cats like tuna, then dogs like beef and pigs like mud.
- (c) If pigs like mud, then so do both dogs and cats.
- (d) Either pigs like mud and cats like tuna, or dogs like beef and cats like tuna.

Answer #1

- Let p be "dogs like beef", q be "cats like tuna", and r be "pigs like mud". Which of the following English statements matches "$(r \vee q) \wedge (p \vee q)$"?
- *(a) Either pigs like mud or cats like tuna, and either dogs like beef or cats like tuna.*
- (b) If cats like tuna, then dogs like beef and pigs like mud.
- (c) If pigs like mud, then so do both dogs and cats.
- (d) Either pigs like mud and cats like tuna, or dogs like beef and cats like tuna.

NOT and XOR

- The **negation** of x is written ¬x, is read "not x", and is true when x is false and false when x is true. In Java the negation operator is !.
- The **exclusive or** of x and y is written x ⊕ y, read "x exclusive or y" or "x or y, but not both", and is true if one of x and y is true and the other false. In Java we can write x ^ y to compute the exclusive or of x and y.

Implication

- The last two boolean operators we will define are **implication** and **equivalence**. These are important in mathematics because each expresses a relationship between propositions that we often want to prove.
- The implication $x \rightarrow y$ is read "if x, then y" or "x implies y". It is true if *either* x is false or y is true. Equivalently, it is true *unless* x is true and y is false. It's important to learn this formal definition, whatever you think "if" means.

Practice Clicker Question #2

- Let p be "frogs are green" and q be "trout live in trees". Which English sentence means the same as "¬(p → ¬q)"?
- (a) It is not the case that if frogs are green, then trout do not live in trees.
- (b) If frogs are not green, then trout do not live in trees.
- (c) If trout live in trees, then frogs are green.
- (d) It is not the case that frogs are green and trout live in trees.

Answer #2

- Let p be "frogs are green" and q be "trout live in trees". Which English sentence means the same as "$\neg(p \rightarrow \neg q)$"?
- *(a) It is not the case that if frogs are green, then trout do not live in trees.*
- (b) If frogs are not green, then trout do not live in trees. $\neg p \rightarrow \neg q$
- (c) If trout live in trees, then frogs are green.
- (d) It is not the case that frogs are green and trout live in trees. $\neg(p \wedge q)$

`false` implies anything

- Normally in mathematics we want to make some **assumptions** and prove that some must be true if the assumptions are true. This is an implication.
- Given our rule, from any false proposition we can prove anything else. Bertrand Russell gave an example of a proof of "I am Elvis" from the premise "0 = 1". ("1 = 2 by arithmetic, Elvis and I are two people, thus Elvis and I are one person".)

Equivalence

- Two boolean values are **equivalent** if they are both true or both false. If x and y are propositions, x ↔ y is the proposition that x and y are equivalent. We can write this in Java as `x == y`.
- We are often interested in the equivalence of two compound propositions with the same atomic propositions. For example, "x → y" and "¬x ∨ y" are equivalent.

More on Equivalence

- How do we know this? They are each true in three of the four possible cases -- they are false only if x is true and y is false. They have the same **truth tables**, as we will soon see.
- As in Java, we have rules for precedence of operations. Negation is first, then the operators $\wedge$, $\vee$, and $\oplus$, then the operators $\rightarrow$ and $\leftrightarrow$. So we can write our equivalence of x $\rightarrow$ y and $\neg x \vee y$ as the single compound proposition $(x \rightarrow y) \leftrightarrow (\neg x \vee y)$.

Tautologies

- Notice a special fact about the compound proposition "$(x \rightarrow y) \leftrightarrow (\neg x \vee y)$". It is true in

  all four possible situations of truth values for x and y, so it is *always true*. We call such a compound proposition a **tautology**.

- In the next lecture we will learn a systematic method to show that a compound proposition is a tautology, by checking all the possible combinations of values of its atomic propositions.

Practice Clicker Question #3

- Which of these statements is *not* a tautology?
- (a) Trout live in trees and trout don't live in trees.
- (b) Trout live in trees or trout don't live in trees.
- (c) If trout live in trees, then trout live in trees.
- (d) Either trout live in trees or trout do not live in trees, but not both.

Answer #3

- Which of these statements is *not* a tautology?
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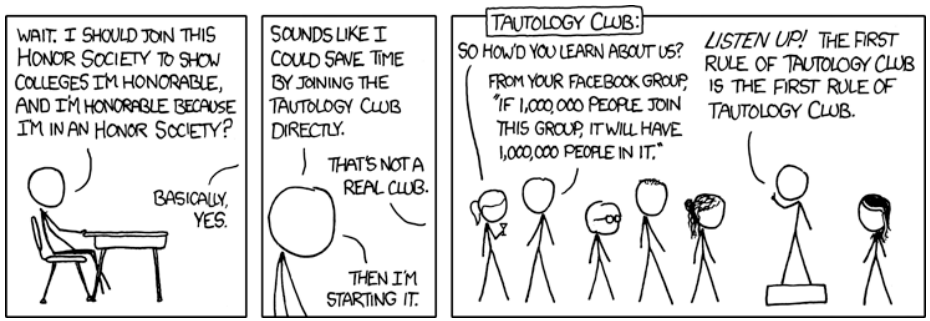
- (b) Trout live in trees or trout don't live in trees.
- (c) If trout live in trees, then trout live in trees.
- (d) Either trout live in trees or trout do not live in trees, but not both.

Obligatory xkcd Reference



xkcd.com/703

The Bigger Picture

- Next week we will see how to use particular tautologies as rules, chaining them together to verify larger tautologies without having to check all the possible cases.
- If there are many atomic propositions, this may be the only feasible way to verify the tautology. Remember that if there are k atomic propositions, there are 2^k possible cases!

The Bigger Picture

- In mathematics, our central task with boolean values turns out to be verifying that particular implications or equivalences *are* tautologies.
- Verifying $x \rightarrow y$ means that if we assume x, we may conclude y.
- Verifying $x \leftrightarrow y$ means that x and y are in effect the same compound proposition.