# CMPSCI 250: Introduction to Computation

Lecture #12: Mathematical Induction
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# Mathematical Induction

- Proving Statements for All Natural Numbers
- Why is Induction Valid?
- Arithmetic and Geometric Sums
- Counting Subsets and Strings
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- Wrong Induction: Elvis is Everybody
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# Proving Statements for All Natural Numbers

- We're now going to learn the important proof technique of **mathematical induction**. Although there are other important forms of it, we will first look at the version that is used to prove a statement of the form $\forall n$:P(n), where the type of n is "natural number". This means that P(0), P(1), P(2), and all other statements of the form P(n) are true. But we don't have time to prove all these statements individually.

- Let's see an example before the formal definition of the technique. Let P(n) be the statement "the sum of the first n positive odd numbers is $n^2$". P(0) says the sum of the first 0 numbers is $0^2$, and this is true. P(1) says that the sum of the first 1 number is $1^2$, and this is true. P(2) says that $1 + 3 = 2^2$, which is again true. P(3) says that $1 + 3 + 5 = 3^2$. But no matter how many instances of P(n) we check, we won't be able to prove $\forall n$:P(n) by just checking instances.

# An Induction Proof

- The induction proof of $\forall n{:}P(n)$ begins with the **base case**. We must prove the statement P(0), which we just did by observing that a sum of no numbers must be 0, and P(0) says that this sum is supposed to be $0^2$. Base cases are usually, but not always, easy to prove.

- Next we must do the **inductive step**. We let n be arbitrary and *assume* that the statement P(n) is true -- P(n) is now called the **inductive hypothesis**. So we are assuming that the sum of the first n odd numbers is $n^2$. Now we have to prove that P(n+1) is true, and P(n+1) says that the sum of the first n + 1 odd numbers is $(n+1)^2$. Clearly the sum of the first n + 1 odd numbers is the sum of the first n odd numbers, added to the n+1'st odd number. What's the n+1'st odd number? It is 2n +1 (we can convince ourselves of this easily, though formally we could do another induction proof). That means that *given our inductive hypothesis*, the sum of the first n + 1 odd numbers is $n^2$ (for the first n) + (2n + 1) which is $n^2 + 2n + 1$ which equals $(n + 1)^2$. We have proved P(n+1), which completes the inductive step and completes the proof.

# Formal Definition of Induction

- The **Law of Mathematical Induction** says that given the two statements P(0) and ∀n:P(n) → P(n+1), we may derive the statement ∀n:P(n).

- An induction proof thus breaks down into the steps we just saw: (1) prove P(0), the **base case**, (2) let n be arbitrary and assume P(n), the **inductive hypothesis**, (3) prove P(n+1) using this hypothesis, the **inductive step**.

- It's a bit of a strange idea to prove a ∀n statement by proving a *more complicated* ∀n statement. But in fact the inductive step is often easier because we have the inductive hypothesis as a premise when we set out to prove P(n) → P(n+1).

- Induction also looks like circular reasoning, in that we assume P(n) on the way to proving ∀n:P(n). But it is the fact that we can get from P(n) to P(n+1) that makes the induction proof valid.

# iClicker Question #1: Parts of an Inductive Proof

- Suppose I want to prove that for any natural number n, the number $n^2 + n$ is even. The statement "$0^2 + 0$ is even" would be **what part** of the proof?
- (a) The base case
- (b) The inductive hypothesis
- (c) The inductive step
- (d) The conclusion

# Why is Induction Valid?

- In 1889 Giuseppe Peano formulated a set of axioms for the natural numbers, which we can paraphrase as follows:

  (1) 0 is a natural number
  (2) Every natural number has a unique successor which is a natural number.
  (3) 0 is not the successor of any natural number.
  (4) Every natural number except 0 is the successor of exactly one natural number.
  (5) Any set of natural numbers that contains 0, and is closed under successor, is the set of all natural numbers.

- The last axiom tells us that mathematical induction works, because we prove that the set $S = \{n: P(n) \text{ is true}\}$ both contains 0 and is closed under successor. Then the axiom tells us that all natural numbers are in S.

- When we prove $\forall n{:}P(n)$ by induction, we also prove that for any particular natural k, there is a chain of implications $P(0) \rightarrow P(1) \rightarrow P(2) \rightarrow \ldots \rightarrow P(k)$ that we could eventually use to prove P(k) from P(0) by Modus Ponens k times.

## Summing the First n Positive Numbers

- You have probably learned somewhere that the sum of the first n positive numbers is n(n+1)/2 -- let's prove this. Let S(n) denote the sum of the first n numbers, and let P(n) be the statement "S(n) = n(n+1)/2". (Note, not for the last time, that P(n) is a *boolean* statement, not a number.)

- For the base case, P(0) says that S(0) = 0(0+1)/2 = 0, which is true.

- We assume that S(n) = n(n+1)/2. By the definition of S, S(n+1) = S(n) + (n+1). Applying the inductive hypothesis, S(n+1) = n(n+1)/2 + (n+1). By algebra, this is (n+1)(n/2 + 1) = (n+1)(n+2)/2, which is just what P(n+1) says it should be.

- We have completed the inductive step and thus completed the proof.

## Sums of Arithmetic Progressions

- An **arithmetic progression** $a_0, a_1, a_2, \ldots, a_n$ is a sequence of numbers where each number $a_{i+1}$ is equal to $a_i + c$, for some constant c. Let P(n) be the statement that the sum $a_0 + \ldots + a_n$ is equal to $(n+1)(a_0 + a_n)/2$.

- The base case P(0) says that the sum $a_0$ equals $(0+1)(a_0 + a_0)/2$, which is true.

- Let S(n) be the sum $a_0 + \ldots + a_n$, so that $S(n+1) = S(n) + a_{n+1}$. The inductive hypothesis P(n) says that $S(n) = (n+1)(a_0 + a_n)/2$, so if we assume P(n) we have that $S(n+1) = (n+1)(a_0 + a_n)/2 + a_{n+1}$. Let's write $a_n$ as $a_{n+1} - c$, so that we get $S(n+1) = (1/2)((n+1)a_0 + (n+1)(a_{n+1} - c) + 2a_{n+1})$. Write one of the two $a_{n+1}$'s as $a_0 + (n+1)c$ and leave the other one alone -- we get $(1/2)((n+1)a_0 + (n+1)a_{n+1} - (n+1)c + a_0 + (n+1)c + a_{n+1}) = (1/2)((n+2)a_0 + (n+2)a_{n+1})$. This is just what the statement P(n+1) says it should be, so we have completed the inductive step and completed the proof.

# Sums of Geometric Progressions

- A geometric progression $a_0, ..., a_n$ is a sequence of numbers where each $a_{i+1}$ is equal to $ra_i$ for some constant r, so that $a_i = a_0r^i$. Let's prove that the sum $a_0 + ... + a_n = (a_0 - a_{n+1})/(1 - r)$, assuming that $r \neq 1$. Again let S(n) be the sum, and let P(n) be the statement that S(n) has the correct value.

- For the base case, P(0) says that S(0) (which is just $a_0$) is $(a_0 - a_1)/(1 - r)$ which is true because $a_1 = ra_0$.

- Assume as IH that $S(n) = (a_0 - a_{n+1})/(1 - r)$. Then S(n+1) is defined to be S(n) + $a_{n+1}$ which by the IH is $(a_0 - a_{n+1})/(1 - r) + a_{n+1} = (a_0 - a_{n+1} + (1 - r)a_{n+1})/(1 - r) = (a_0 - a_{n+1} + a_{n+1} - ra_{n+1})/(1 - r) = (a_0 - a_{n+2})/(1 - r)$ because $a_{n+2} = ra_{n+1}$. This is just what the statement P(n+1) says that S(n+1) should be, so we have completed the inductive step and completed the proof.

# iClicker Question #2: Proving an Inequality

- Let P(n) be the statement "$n! \leq n^n$". (Remember that "n!", called "**n factorial**", is the product $1 \cdot 2 \cdot \ldots \cdot n$.) If we want to prove this statement by induction, **what should be our inductive step**?

- (a) "Observe that $0! \leq 0^0$, since both empty products are 1."

- (b) "Assume that $n! \leq n^n$."

- (c) "Given that $n! \leq n^n$, prove that $(n+1)! \leq (n+1)^{n+1}$."

## Counting Subsets and Strings

- We learned earlier in the course that a set with k elements has exactly $2^k$ subsets. (That is, the **power set** of a k-element set has exactly $2^k$ subsets in it.) We can prove this statement by induction as well. Let P(n) be the statement "any n-element set has exactly $2^n$ subsets".

- For the base case, the empty set has exactly one subset (itself), and $2^0 = 1$.

- Assume as IH that any n-element set has exactly $2^n$ subsets. Let S be an arbitrary n+1-element set, and note that S is $T \cup \{x\}$ where T is an n-element set and x is some element. By the IH, T has exactly $2^n$ subsets. Each of these subsets Z gives rise to *exactly two* subsets of S, Z and $Z \cup \{x\}$. And every subset of S must be either a subset of T, or a subset of T with x unioned in. So the number of subsets of S is exactly 2 times $2^n$, or $2^{n+1}$, just as P(n+1) says it should be. This completes the inductive step and thus also the proof.

# iClicker Question #3: Counting Strings

- An almost identical induction proof tells us that there are exactly $2^n$ binary strings of length n. **What is the inductive hypothesis** of this proof?

- (a) “Assume that there is exactly one binary string of length 0.”

- (b) “If there are $2^n$ binary strings of length n, there must be $2^{n+1}$ binary strings of length n + 1.”

- (c) “Assume that there are exactly $2^n$ binary strings of length n.”

# Creative Induction: Odd Pie Fights

- Here's a somewhat different use of induction to prove a geometric result. We're going to stage a pie fight by placing an *odd number* of people in the plane, in such a way that every pair of people has a distinct distance between them. At a signal, each person will throw a pie at the closest other person. Although n total pies are thrown at n people, we will show that at least one person does not get hit with a pie.

- Let P(k) be the statement "in such a fight with 2k + 1 people, at least one person is not hit". P(0) is true because with one person, there is no one else to throw a pie at them and they are not hit. Assume that P(k) is true and look at a pie fight with 2k + 3 people. One pair of people, whom we may call x and y, have the shortest distance of any pair and thus throw pies at each other. Now look at the other 2k + 1 people. If none of them throw at x or y, they are in a 2k + 1 person pie fight and the IH says that one of them is not hit. But if they *do* throw at x or y, they are just removing pies from a 2k + 1 person pie fight and there is still a person not hit. (Everyone of the other 2k + 1 people who does not throw at x or y throws at their closest person among the others.)

# Wrong Induction: Elvis is Everybody

- Here is an **invalid proof** that *you* are the late Elvis Presley, in fact that everyone is Elvis. We let P(n) be the statement "in any set of n people that includes an Elvis, everyone is Elvis". The base case P(0) is **vacuous** as there can be no such set, and the case P(1) is obviously true.

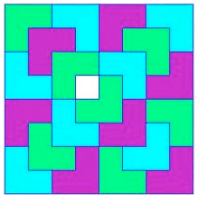
- Assume the IH of P(n) and let S be any set of n + 1 people containing an Elvis named e. Let x be some element of S other than e, and let $T = S \setminus \{x\}$. Let U be the set $S \setminus \{e\}$. T is a set of n people containing an Elvis (it contains e), so every member of T is an Elvis. U is also a set of n people, and it contains an Elvis because we just proved that all the people in $T \cap U$ are Elvises. So every member of U is an Elvis, and thus every member of S is an Elvis and we win.

- The problem with this proof is the n = 1 case. If S has exactly two people, there are no "people in $T \cap U$", and the inference that x is an Elvis is not valid.

# iClicker Question #4: Looking at the Elvis Proof

- Again, let P(n) be the statement "In any group of n people containing an Elvis, all are Elvises." **For what values of n** is this statement actually true?
- (a) for no values at all
- (b) only for n = 0
- (c) only for n = 0 and n = 1
- (d) only for n = 0, n = 1, and n = 2

## Tilings Again



- We looked at tilings by dominoes (1 by 2 rectangles) and straight triominoes (1 by 3 rectangles). What about covering a chessboard with **L-shaped triominoes**? Of course we have to leave out a square, so that we can cover 63 squares with 21 pieces. The figure at right shows that if the missing square is near the middle, we can do it. Can we do it for every missing square?

- The answer is an interesting use of induction. Let P(n) be the statement "any $2^n$ by $2^n$ board, with any one square missing, can be tiled with L-shaped triominoes". P(0) talks about a 1 by 1 board with one square missing, which we can tile with no pieces. P(1) talks about a 2 by 2 with one square missing, which we can tile with one piece. Our problem above is P(3). Can we prove $P(n) \rightarrow P(n+1)$?

## The Inductive Step for Tilings

- Assume that P(n) is true, so that we can tile any $2^n$ by $2^n$ board with any one square missing. We want to prove P(n+1), which says that we can tile a $2^{n+1}$ by $2^{n+1}$ board with any one square missing.

- Consider such a board, and divide it into four $2^n$ by $2^n$ subboards. *One* of the subboards has a square missing, because the missing square from the big board is in one of the four. Place one L-shaped piece at the middle of the big board, so as to take one square away from each of the *other three* subboards. Now we have four subboards, each $2^n$ by $2^n$ with one square missing. The inductive hypothesis says that each of these subboards can be tiled with L-shaped triominoes. So we have tiled the whole original board, except for the missing square, and we have thus proved the desired statement P(n+1). By induction, P(n) must be true for all n.

- If we wanted to tile a 1024 by 1024 board with one square missing, this proof actually gives us a **recursive algorithm** to do it.