CMPSCI 250 Discussion #9: Shortest Paths With Matrices

Individual Handout
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The **all-pairs shortest path problem** takes a labeled directed graph as input. If there is an edge from vertex u to vertex v with label d, that means that you can go from u to v in one step in distance v. If there is no edge from u to v, the single-step distance from u to v is ∞. If u and v are the same vertex, the single-step distance is 0. We can give the input in the form of a **single-step distance matrix** as well as in the form of a labeled graph.

The output of the problem is the **all-pairs shortest-path distance matrix**. The (u, v) entry of this matrix gives the distance from u to v on the *shortest possible path.* One way to get this, as we'll see next week, is to do a uniform-cost search from each of the vertices. But matrix multiplication gives us a way to get all the shortest-path distances at once.

Remember that the general form of the Path-Matrix Theorem says that if A is the matrix giving the label of each edge of a directed graph, then A^t for any natural number t has the following meaning. The $(u, v,)$ entry of A^t gives the "sum", over all t-step paths from u to v, of the "product" of labels of the edges on the path. This works for any definition of "sum" and 'product" that obeys the usual rules of arithmetic for those operations.

With ordinary natural-number sum and product, when A_{uv} was the number of arcs from u to v, A^t_{uv} was the number of t-step paths from u to v. Now we redefine "sum" to be the **minimum** of the arguments, and redefine "product" to be the **sum** of the arguments.

Question 1: Show that this "sum" and "product" obey the distributive law, that for any three numbers x, y, and z, "$x(y + z)$" equals "$xy + xz$". That is, translate this equation using the new meaning of "sum" and "product", and show that it is valid.

When we compute A^t, each entry gives the *minimum* over all t-step paths of the *sum* of the labels of the edges on the path. This is just the length of the shortest t-step path. The entries of 0 on the main diagonal mean that we have implicit length-0 edges from each vertex to itself. A nice consequence of this is that when we consider t-step paths, we also get all paths with fewer than t steps because these shorter paths each become t-step paths when the implicit 0-loops are taken into account. When we let t be one less than the number of vertices n in the graph, A^t gives us the length of each shortest path of at most $n - 1$ steps. As long as all our edge labels are non-negative, the shortest path is a simple path, and this gives the true distance.

Here is the matrix we will use for the rest of this exercise:

$$\begin{pmatrix} 0 & 1 & 3 & \infty & 7 \\ \infty & 0 & 1 & \infty & \infty \\ \infty & \infty & 0 & 1 & 3 \\ 4 & 3 & \infty & 0 & 1 \\ \infty & \infty & 4 & 0 & 0 \end{pmatrix}$$

Question 2: Draw the labeled directed graph represented by this matrix, with vertices named a, b, c, d, and e.

Question 3: Compute the entry of the matrix A^2 that represents the shortest two-step path from vertex a to vertex e. This should be the minimum of five sums.

Question 4: Compute the remainder of the matrix A^2 using this "sum" and "product".

Question 5: Compute the entry of the matrix A^4 that represents the shortest four-step path from vertex a to vertex e. Identify this path in the graph. Note that the easiest way to get A^4 is to multiply A^2 by itself.

Question 6: If you have time, calculate as much of the matrix A^4 as you can.