CMPSCI 250: Introduction to Computation

Lecture #21: Induction for Problem Solving
David Mix Barrington
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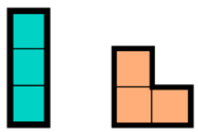
Induction for Problem Solving

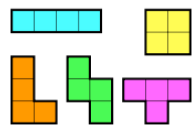
- The L-Shaped Tile Problem
- Recursively Tiling a Chessboard
- Cutting Pizzas
- The Pizza-Cutting Theorem
- Cutting a Block of Cheese With a Katana
- The Speed of the Euclidean Algorithm

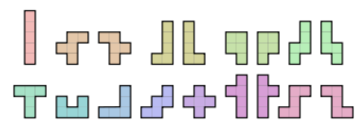
The L-Shaped Tile Problem

- Samuel Golomb initiated the study of generalized dominos called **polyominos**. An ordinary domino is made from two connected squares, and there is basically only one way to do it. A **tromino** is made from three connected squares, and there are two different ones, I-shaped and L-shaped. There are five kinds of **tetrominos** and twelve kinds of **pentominos** (six of which are shown twice below, in two reflective forms).

- Golomb posed the question of what kinds of figures can be **tiled** by various kinds of polyominos. In particular, can an 8×8 chessboard be tiled by L-shaped trominos? (No, because $64 \% 3 \neq 0$.) What about an 8×8 board with *one square missing*?







Figures from Wikipedia articles “Tromino”, “Tetromino”, and “Pentomino”

Recursively Tiling a Chessboard

- In the figure below, an 8×8 board with one (black) square missing has been tiled by 21 L-shaped trominos. How did we do it, and can we always do it?

- We do it with a *recursive algorithm* that provides an *inductive proof* that **any $2^n \times 2^n$ board, with any one square missing, can be tiled**. (The bold-faced statement will be the P(n) of our inductive proof.) The base case of P(0) says that any 1×1 board, with any one square missing, can be tiled. (Use 0 tiles!)

- The key step of the recursive algorithm is to reduce a $2^{n+1} \times 2^{n+1}$ problem to four $2^n \times 2^n$ problems. We do this by placing one tile (the orange one in the figure) to take one square from each quarter of the board, except for the quarter that is already missing a square. Then we recursive tile each of the quarters -- in this example by placing the green tile to make 2×2 boards with one square missing -- these are covered by the red and blue tiles.

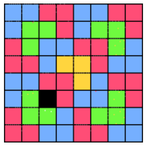
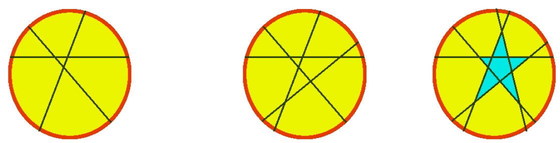


Figure from math.hmc.edu
"Math Fun Facts"

Cutting Pizzas

- Our next problem is to determine the maximum number of pieces into which we can divide a circular pizza, using only straight cuts in the plane of the pizza.
- With no cuts we have one piece, with one cut we can make two, and with two cuts we can make four. We can't ever do better than doubling the number of pieces, because the pieces (by induction) are all convex and a single straight cut can divide a convex piece into at most two subpieces.
- This leaves the possibility of getting eight pieces with three cuts, but that doesn't seem to be possible. You can get six by having all three cuts go through a single point, or seven by having then almost do that but leave a triangle. Here are a cut into 11 pieces with four cuts, and 16 with five cuts:



Diagrams from murderousmaths.co.uk

The Pizza-Cutting Theorem

- **Theorem:** The maximum possible number of pieces with n such cuts is exactly $(n^2 + n + 2)/2$. We prove this theorem by induction on n. The base case says that we have $(0^2 + 0 + 2)/2 = 1$ piece with no cuts, which is true.

- Suppose that we have $(n^2 + n + 2)/2$ pieces with n cuts, and we want to make an n+1'st cut. By geometry, two straight lines meet in at most one point, so the new cut can cross each old cut at most once. This means that the new cut passes through at most n + 1 old pieces. Those pieces are divided into two and the others are not. So after n + 1 cuts we have at most $(n^2 + n + 2)/2 + (n + 1) = (n^2 + 3n + 4)/2 = (n + 1)^2 + (n + 1) + 2$, *at most* the number we want.

- We have to show that this number is always achievable. If the n + 1 cuts are in **general position**, meaning that every two cuts meet and that three or more cuts never meet in the same place, we achieve our bound as long as the pizza is big enough to include all the intersection points.

Cutting a Block of Cheese With a Katana

- A similar analysis works in three dimensions. With a katana (Japanese samurai's sword), we can make straight cuts through a convex block of cheese in any plane we like. (We are not allowed to move the pieces until all cuts have been made.) The first three cuts can be mutually perpendicular, giving us 1 piece with no cuts, 2 pieces with 1, 4 pieces with 2, and 8 with 3.

- The fourth cut is hard to visualize, but if your cut goes *near but not through* the existing triple intersection point, you cut seven of the eight old pieces and get a new total of 15. With a fifth cut, you can get 26 pieces, a number rather hard to figure out by visualizing (but see a way that a UMass alum got the number, at least, at www.gweep.net/~sskoog/block_of_cheese_problem.txt).

- The key to the solution is how many new pieces we add with each cut. From the above, the sequence is 1, 2, 4, 7, 11,... which we might recognize as the number of pieces of pizza in the previous problem. Why is this?

The Cheese Cutting Theorem

- **Theorem:** The maximum number of cheese pieces is exactly $(n^3 + 5n + 6)/6$.

- Where did this polynomial come from? It is the only function of the form $f(n) = an^3 + bn^2 + cn + d$ that satisfies $f(0) = 1$, $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. Since the difference function is a quadratic, we expect this to be a cubic.

- The n+1'st katana cut forms a new plane within the cheese, and the intersection of each of the old cuts is a straight line. These n lines divide our new plane into at most $(n^2 + n + 2)/2$ regions by the Pizza Cutting Theorem. If we compute $(n^3 + 5n + 6)/6 + (n^2 + n + 2)/2$ we get $(n^3 + 3n^2 + 8n + 12)/6$ which is exactly $((n + 1)^3 + 5(n + 1) + 6)/6$, the number we want as the conclusion of our inductive step.

- But if the cuts are in general position, and the block of cheese is large enough to include all the intersections, we achieve exactly this number.

The Speed of the Euclidean Algorithm

- We asserted that the Euclidean Algorithm can quickly test even very large numbers for relative primality. For example, 2068 and 1259 give 809, 450, 359, 91, 86, 5, 1, and 2068 and 1289 give 779, 510, 269, 241, 28, 17, 11, 6, 5, and 1.

- Consecutive Fibonacci numbers take a relatively long time, e.g., 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, 1, but we know from last week's discussion that F(n) is about $(1.61)^n$, so that if x is a Fibonacci number we take about $\log_{1.61} x$ steps.

- Here we'll show that if the two initial numbers are each at most 2^n, the EA will terminate in at most 2n + 1 steps. The base case of our induction says that if both numbers are at most $2^0 = 1$, we need 2(0) + 1 = 1 step.

- The inductive step uses the contrapositive method. We start with a and b, and compute a = qb + c and b = rc + d, so a = (qr + 1)c + qd. If c or d is *greater than* 2^n, then a is greater than 2^{n+1}. So if $a \leq 2^{n+1}$, then $c \leq 2^n$. By the IH we need at most 2n + 1 steps starting with c and d, so at most 2n + 3 total.