CMPSCI 250: Introduction to Computation

Lecture #11: Equivalence Relations
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Equivalence Relations

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Definition of an Equivalence Relation

- Last lecture we looked a partial orders, which are reflexive, antisymmetric, and transitive. Today we look at **equivalence relations**: binary relations on a set that are reflexive, symmetric, and transitive.

- Recall the definitions: R is **reflexive** if $\forall x: R(x, x)$, R is **symmetric** if $\forall x: \forall y: R(x, y) \rightarrow R(y, x))$, and R is **transitive** if $\forall x: \forall y: \forall z: (R(x, y) \wedge R(y, z)) \rightarrow R(x, z)$.

- You should be familiar with these properties of the **equality relation**: "$x = x$" is always true, from "$x = y$" we can get "$y = x$", and we know that if $x = y$ and $y = z$, then $x = z$. The idea of equivalence relations is to formalize the property of *acting like equality in this way*.

- To prove that a relation is an equivalence relation, we formally need to use the Rule of Generalization, though we often skip steps if they are obvious.

Two More Examples: Universal and Parity

- If A is any set, we can define the **universal relation** U on A to *always be true.* Formally, U is the entire set $A \times A$ consisting of all possible ordered pairs.

- Of course U(x, x) is always true, and the implications in the definitions of symmetry and transitivity are always true because their conclusions are true.

- The *always false* relation $\neg U$ is symmetric and transitive but not reflexive.

- The **parity relation** on naturals is perhaps more interesting. We define P(i, j) to be true if i and j are either both even or both odd. Later we'll call this "being congruent modulo 2" and define being congruent modulo n in general.

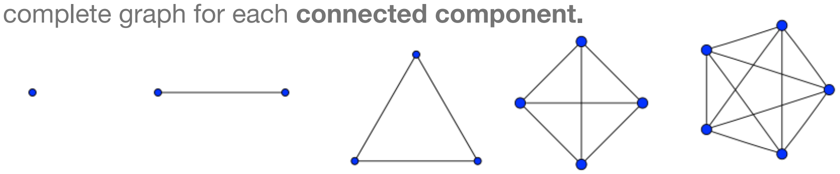
- Any relation of the form "x and y are the same in this respect" will normally be reflexive, symmetric, and transitive, and thus an equivalence relation.

The Graph of an Equivalence Relation

- What happens when we draw the diagram of an equivalence relation?

- Because it is reflexive, we have a loop on every vertex, but we can leave those out for clarity. The arrows are bidirectional because the relation is symmetric.

- If we have a set of points that have *some* connection from each point to each other point, transitivity forces us to have *all possible direct connections* among those points. A graph with all possible undirected edges is called a **complete graph** on its points. The graph of an equivalence relation has a complete graph for each **connected component.**



Complete graphs for up to five points, from wikipedia.com “Complete Graph”

Partitions and the Partition Theorem

- Let's prove that this characterization of the graph is correct -- we will need a new definition.

- If A is any set, a **partition of** A is a *set of subsets* of A -- a set $P = \{S_1, S_2, \ldots, S_k\}$ where (1) each S_i is a subset of A, (2) the union of all the S_i's is A, and (3) the sets are **pairwise disjoint** -- $\forall i: \forall j: (i \neq j) \rightarrow (S_i \cap S_j = \varnothing)$.

- The **Partition Theorem** relates equivalence relations to partitions. It says that a relation is an equivalence relation if and only if it is the "same-set" relation of some partition. In symbols, the same-set relation of P is given by the predicate SS(x, y) defined to be true if $\exists i: (x \in S_i) \wedge (y \in S_i)$.

- So we need to get a partition from any equivalence relation, and an equivalence relation from any partition.

“Same-Set” on a Partition is an E. R.

- Let P = {S_1, S_2,..., S_k} be a partition of A and let SS be its same set relation.

- We first show that SS is reflexive. Let x be an arbitrary element of A. Because the sets of P union to give A, x must be in at least one of them, S_i. So $(x \in S_i) \wedge (x \in S_j)$ is true, and thus SS(x, x) is true for an arbitrary x.

- To show SS is symmetric, let x and y be arbitrary elements of A and assume that SS(x, y) is true. We need to prove SS(y, x). But we have $(x \in S_i) \wedge (y \in S_i)$ from the definition, and we can rewrite this as $(y \in S_i) \wedge (x \in S_i)$ and get SS(y, x).

- For transitivity, we let x, y, and z be arbitrary and assume SS(x, y) and SS(y, z). From the definition we know that x and y are both in some S_i and that y and z are both in some S_j. But since y is in both S_i and S_j, and the sets are pairwise set, the sets S_i and S_j are the same, and this single set contains both x and z. So SS(x, z) is true, and we have proved that SS is transitive.

Equivalence Classes

- If R is an equivalence relation on A, and x is any element of A, we define the **equivalence class** of x, written [x], as the set {y: R(x, y)}, that is, the set of elements of A that are related to x by R.

- The universal relation U has a single equivalence class consisting of all the elements. The equality relation has a separate equivalence class for each element.

- In the parity relation, the set of even numbers forms one equivalence class and the set of odd numbers forms another.

- If we let A be the set of people in the USA, and define R(x, y) to mean “x and y are legal residents of the same state”, we get fifty equivalence classes, one for each state. One of them is {x: x is a legal resident of Massachusetts}.

The Classes Form a Partition

- To finish the proof of the Partition Theorem, we must prove that if R is any equivalence relation on A, the set of equivalence classes forms a partition.
- Note that in the *set* of classes, we only count a class once even if it has multiple definitions. So if [x] and [y] are the same set, it is just one set of the partition.
- Recall our three conditions for a set of sets to be a partition. Condition (1) says that each set is a subset of A, which is clearly true for the classes.
- Condition (2) says that the sets union together to give A, which is true for the classes because each element is in at least one class, its own.
- We still have to show (3) for the classes, that they are pairwise disjoint.

Finishing the Proof

- Let [x] and [y] be the equivalence classes of two arbitrary elements x and y of A. (This gives us two arbitrary equivalence classes, which might or might not be equal as sets.)

- We must show that $([x] \neq [y]) \rightarrow ([x] \cap [y] = \emptyset)$. We'll do this by contrapositive, showing $(\exists z: z \in [x] \cap [y]) \rightarrow ([x] = [y])$.

- Assume that an element z of $[x] \cap [y]$ exists and name it z. We must show that $[x] = [y]$, which means $\forall w: (w \in [x]) \leftrightarrow (w \in [y])$. By the definition of equivalence classes, this means $\forall w: R(x, w) \leftrightarrow R(y, w)$. So let w be arbitrary.

- We know that R(x, z) and R(y, z). Assume R(x, w). We have R(z, x) by symmetry, and then R(y, z), R(z, x), and R(x, w) give us R(y, w) by transitivity.

- The argument that $R(y, w) \rightarrow R(x, w)$ is exactly the same as $R(x, w) \rightarrow R(y, w)$.