1. Let $P(n)$ be the statement that $n! < n^n$, where $n \geq 2$ is an integer.

   Basis step: $2! = 2 \cdot 1 = 2 < 4 = 2^2$

   Inductive hypothesis: Assume $k! < k^k$ for some $k \geq 2$.

   (We need to show that $P(k+1)$ is true, given the inductive hypothesis.)

   Inductive step:

$$\begin{aligned}(k+1)! &= (k+1)k! \\ &< (k+1)k^k \\ &< (k+1)(k+1)^k \\ &= (k+1)^{k+1}\end{aligned}$$

   Now, since we have completed the base and inductive steps, by the principle of mathematical induction, the inequality is true for any $n \geq 2$. If we had shown $P(3)$ as our basis step, then the inequality would only be proven for $n \geq 3$.

2. For any positive integer n

$$\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

   Proof by induction on n.

   Basis step: Let $n = 1$. Then

$$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}.$$

   Inductive hypothesis: Assume that for some positive integer k

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

Inductive step:

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2}
\end{aligned}$$

□

3. For any positive integer n

$$\sum_{i=1}^{n} i \cdot i! = 1\cdot 1! + 2\cdot 2! + \ldots + n\cdot n! = (n+1)! - 1$$

Proof by induction on n.

Basis step: Let $n = 1$. Then

$$\sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1$$

and

$$(1+1)! - 1 = 2! - 1 = 2 - 1 = 1.$$

Inductive hypothesis: Assume that for some positive integer k

$$\sum_{i=1}^{k} i \cdot i! = 1\cdot 1! + 2\cdot 2! + \ldots + k\cdot k! = (k+1)! - 1$$

Inductive step:

$$\begin{aligned}\sum_{i=1}^{k+1} i \cdot i! &= 1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \\ &= (k+1)!(1 + (k+1)) - 1 \\ &= (k+2)(k+1)! - 1 \\ &= (k+2)! - 1\end{aligned}$$

□

4. For any positive integer n

$$\sum_{i=1}^{n} i(i+1)(i+2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4$$

Proof by induction on n.

Basis step: Let $n = 1$. Then

$$\sum_{i=1}^{1} i(i+1)(i+2) = 1 \cdot 2 \cdot 3 = 6$$

and

$$1(1+1)(1+2)(1+3)/4 = 1 \cdot 2 \cdot 3 \cdot 4/4 = 6$$

Inductive hypothesis: Assume that for some positive integer k

$$\sum_{i=1}^{k} i(i+1)(i+2) = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) = k(k+1)(k+2)(k+3)/4$$

Inductive step:

$$\begin{aligned}\sum_{i=1}^{k+1} i(i+1)(i+2) &= 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) \\ &= k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3) \\ &= (k+1)(k+2)(k+3)(k/4 + 1) \\ &= (k+1)(k+2)(k+3)(k/4 + 4/4) \\ &= (k+1)(k+2)(k+3)(k+4)/4\end{aligned}$$

□

5. For any nonnegative integer n, 6 divides $n^3 - n$.

Proof by induction on n.

Basis step: Let $n = 0$. Then $n^3 - n = 0^3 - 0 = 0$, which is divisible by every integer, including 6.

Inductive hypothesis: Assume for some nonnegative integer k that $k^3 - k$ is divisible by 6.

Inductive step:

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= (k^3 - k) + 3k^2 + 3k + 1 - 1 \\ &= (k^3 - k) + 3(k^2 + k)\end{aligned}$$

By the inductive hypothesis, $(k^3 - k)$ is divisible by 6. Clearly, $3(k^2 + k)$ is divisible by 3. To show that it is divisible by 6, it suffices to show that $k^2 + k$ is even. We do this by cases.

Case 1: k is even, which means there exists some integer m such that $k = 2m$, so $k^2 + k = 4m^2 + 2m = 2(2m^2 + m)$ is even.

Case 2: k is odd, which means there exists some integer m such that $k = 2m - 1$, so

$$k^2 + k = (2m-1)^2 + 2m - 1 = 4m^2 - 4m + 1 + 2m - 1 = 4m^2 - 2m = 2(2m^2 - m)$$

is even.

□

6. If n is an integer where $n \geq 3$, then $n^2 - 7n + 12$ is nonnegative.

Proof by induction on n.

Basis step: Let $n = 3$. Then

$$n^2 - 7n + 12 = 3^2 - 7 \cdot 3 + 12 = 9 - 21 + 12 = 0.$$

Inductive hypothesis: Assume for some integer $k \geq 3$ that $k^2 - 7k + 12$ is nonnegative.

Inductive step:

$$\begin{aligned}(k+1)^2 - 7(k+1) + 12 &= k^2 + 2k + 1 - 7k - 7 + 12 \\ &= (k^2 - 7k + 12) + (2k + 1 - 7) \\ &\geq 0 + 2k + 1 - 7 \\ &= 2k - 6 \\ &\geq 2 \cdot 3 - 6 \\ &= 0\end{aligned}$$

□

7. For any nonnegative integer n where $n \neq 2$ and $n \neq 3$, the inequality $n^2 \leq n!$ is true.

Proof. Note first that:

- if $n = 0$, then $0^2 = 0$ and $0! = 1$.
- if $n = 1$, then $1^2 = 1$ and $1! = 1$.
- if $n = 2$, then $2^2 = 4$ and $2! = 2$.
- if $n = 3$, then $3^2 = 9$ and $3! = 6$.

We prove by induction on n that $n^2 \leq n!$ for all $n \geq 4$.

Basis step: $4^2 = 16$ and $4! = 24$

Inductive hypothesis: Assume for some integer $k \geq 4$ that $k^2 \leq k!$.

Inductive step:

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&\geq (k+1)k^2 \\
&= k^2 \cdot k + k^2 \\
&\geq 4^2 \cdot k + k^2 \\
&= 15k + k + k^2 \\
&\geq 15k + 1 + k^2 \\
&\geq 2k + 1 + k^2 \\
&= (k+1)^2
\end{aligned}$$

□