NAME: ______________________

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INFO 150
A Mathematical Foundation for Informatics
SOLUTIONS to Final Exam Fall 2024

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 120 total points. The scale is A=105, B = 87, C=69, D = 51, F = 33.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer. But there may be times when you need to compare numerical answers to one another, which would mean computing them or at least estimating them.

1	/15
2	/20
3	/25
4	/20
5	/20
6	/20
Total	/120

Question 1 (15): Briefly identify and distinguish the following terms or concepts (3 points each):

- (a) a **tautology** and a **contradiction** in boolean logic

  A tautology is a statement that is always true, no matter the values of the inputs. A contradiction is a statement that is always false.

- (b) an **onto function** and a **one-to-one function**

  A function is onto if for every element of the domain, there is at least one element of the domain mapping to it. A function is one-to-one if no two elements of the domain map to the same element of the codomain.

- (c) the **probability** of an event and the **number of ways** an event could occur

  The probability of an event is a number, in the range from 0 through 1. If there are a set of equally likely atomic events, the probability of an event is the number of atomic events in which the event occurs, divided by the total number of atomic events.

- (d) a **directed graph** and an **undirected graph** in graph theory

  Both are sets of nodes and edges, but a directed graph has directed edges, where each edge is *from* one node *to* some node, while an undirected graph has undirected edges, where each edge is *between* two nodes or from one node to itself.

- (e) a **walk** and a **path** in an undirected graph

  A walk in a graph is an alternating sequence of nodes and edges, where each edge is from the vertex before it to the vertex after it. A path is a walk in which no node is used more than once.

Question 2 (20): Seven women from New England state universities are about to compete in a cross-country footrace. They will finish in order from 1 through 7 – there are no ties. The competitors, with their states, are Abigail (MA), Brittany (CT), Carmen (RI), Debra (MA), Enid (MA), Francine (CT), and Gretchen (VT). We define a function f from the set $R = \{A, B, C, D, E, F, G\}$ to the set $P = \{1, 2, 3, 4, 5, 6, 7\}$, with the rule being that $f(X) = n$ if competitor X finishes in position n.

- (a, 5) In how many ways could the race come out? (For example, one of the ways you are counting is "$f(A) = 3$, $f(B) = 7$, $f(C) = 2$, $f(D) = 4$, $f(E) = 1$, $f(F) = 6$, and $f(G) = 5$".

  There are $P(7, 7) = 7! = 5040$ ways to do this.

- (b, 5) In how many of those possible outcomes does Abigail finish third, *and* Enid finish either fourth or fifth, *and* Francine finish first?

  First assume that $f(A) = 3$ and $f(F) = 1$. If Enid finishes fourth, there are $4!$ ways to place the four unassigned competitors in the four remaining places. If Enid finishes fifth, there are $4!$ way for those four unassigned people to go in those remaining places. By the Sum Rule (since Enid cannot finish in both fourth and fifth), we have $4! + 4! = 48$ ways to do this.

- (c, 5) If each possible order of finish were equally likely, what is the the probability that the three Massachusetts competitors would be the top three finishers? (For example, this would happen if the order were "D, A, E, G, F, B, C".)

  Out of the 5040 finishes, this happen in $3!4! = 144$ possible ways, because the three Massachusetts people could finish in any order among their three positions, and the other four could finish in any order among their four positions. So the probability would be $144/5040 = 1/35$.

- (d, 5) Let's consider a "team result" for this race where we list the *states* of each finisher rather than their names. (For example, the team result for the race result example in part (c) would be "MA, MA, MA, VT, CT, CT, RI". How many different team results are possible for this race?

  This the number of anagrams of the word $MMMVCCR$, which we had two ways to computer in the text and lecture. We could find the $C(7, 3)$ ways to place the M, among the seven places, then the $C(4, 2)$ ways to place the two C's among the four remaining places, then $C(2, 1)$ ways to place the V among the two remaining places, then $C(1, 1)$ ways to place R in the one remaining place. This would be $C(7, 3)C(4, 2)C(2, 1) = 35 \cdot 6 \cdot 2 = 420$. We could also use the multinomial formula to get $7!/(4!2!1!1!) = 5040/(6 \cdot 2 \cdot 1 \cdot 1) = 5040/12 = 420$.

Question 3 (25): These translations and proof deal with the scenario in Question 2. Remember that if X finishes *before* Y, then $f(X) < f(Y)$.

- (a, 5) Write a symbolic statement meaning "Enid placed fourth, and Gretchen finished before both Debra and Francine."

  The symbolic statement is "$(f(E) = 4) \wedge (f(G) < f(D)) \wedge (f(G) < f(F))$**".**

  For both (a) and (b), I took off a point for saying, for example, "$f(G) < f(D) \wedge f(F)$*", since the* $\wedge$ *operation is not defined on anything except booleans.*

- (b, 5) Write a symbolic statement meaning "Some person finished after Enid but before both Debra and Francine."

  The symbolic statement is "$\exists X \in R, (f(E) < f(X)) \wedge (f(X) < f(D)) \wedge (f(X) < f(F))$**".**

- (c, 5) Translate into English: $((f(A) = 1) \wedge (f(C) = 3) \wedge (f(G) < f(B))) \rightarrow (f(G) = 2)$.

  "If Abigail finished first, Carmen finished third, and Gretchen finished before Brittany, then Gretchen finished second."

  Several people put the "if" statement in the wrong place.

- (d, 10) Explain carefully why if the statements in part (a) and part (b) are both true, the statement in (c) must also be true. You may quote facts from the lecture and book if you do so clearly.

  Assume that (a) and (b) are true, and that the premise of (c) is also true. We have $f(A) = 1$**,** $f(C) = 3$**, and** $f(E) = 4$**. So** B**,** D**,** F**, and** G **are in positions** 2**,** 5**,** 6**, and** 7**. We know that person** X **is after** E **and before** D **and** F**.** X **has to be in position** 5**,** 6**, or** 7**, and** D **and** F **are still among those three positions, so** $f(X) = 5$**. If** G **were** X**, we could not have** $f(G) < f(B)$**, since all the positions after** X **have been determined and they don't include** G**. So** $f(G)$ **has to be** 2**.**

  The most common error was to assume that Gretchen could not be the person referred to in part (b). She can't be, but you need to use the fact that Gretchen finished before Brittany to prove that.

Question 4 (20): In this problem we define a function G from positive integers to positive integers, defined recursively. It uses the rules $G(1) = 1$, $G(2) = 4$, $G(3) = 9$, and for all n with $n \geq 4$, $G(n) = G(n-1) - G(n-2) + G(n-3) + 4n - 6$. Here we ask you to prove, by induction on n, that for every positive natural n, $G(n)$ is equal to n^2.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

  $P(n)$ **says that the sum** $G(n) = n^2$**.**

  It doesn't say "for all n, $G(n) = n^2$". The statement $P(n)$ refers to only one number at a time.

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

  We need to verify $P(1)$**,** $P(2)$**, and** $P(3)$**, since the inductive rule is only useful for** $P(n)$ **with** $n > 3$**. Here** $G(1) = 1 = 1^2$**,** $G(2) = 4 = 2^2$**, and** $G(3) = 9 = 3^2$**, so all three base cases are correct.**

  Many people also proved $P(4)$, which was fine, as long as you proved the first three.

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

  The IH says that for all i **with** $1 \leq m - 1$**,** $P(i)$ **is true. (In this case we will only need the statements** $P(m-1)$**,** $P(m-2)$**, and** $P(m-3)$**.) The IG says that** $P(m)$ **is true, meaning that** $G(m) = m^2$**.**

  The IG could be $P(m)$ or $P(m+1)$, as long as your IH and IG match. But your IH has to have all three of the statements you need to prove the IG.

- (d, 8) Prove your inductive step, completing the proof.

  We need an inductive step for every m **with** $m \geq 4$**. We need to evaluate** $G(m)$**, and the rule tells us that it is** $G(m-1) - G(m-2) + G(m-3) + 4m - 6$**. By the IH applied to** $m - 1$**,** $m - 2$**, and** $m - 3$**, we get that** $G(m)$ **is equal to** $(m-1)^2 - (m-2)^2 + (m-3)^3$**, which is** $(m^2 - 2m + 1) - (m^2 - 4m + 4) + (m^2 - 6n + 9) + 4m - 6$**. Collecting terms, we have** $(m^2 - m^2 + m^2) + (-2m + 4m - 6m + 4m) + (1 - 4 + 9 - 6) = m^2$**. This completes the induction and thus the proof.**

Question 5 (20): In a particular dice game, you roll four fair six-sided dice, so that each number comes from the set $\{1, 2, 3, 4, 5, 6\}$ with equal probability.

- (a, 5) What is the probability that the four numbers you rolled are all different?

  There are $6^4 = 1296$ for the four numbers to come up, and there are $P(6, 4) = 6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways for you to get four different numbers from the six possibilities. The probability is thus $360/1296 = 5/18$.

- (b, 5) What is the probability that you rolled at least one six?

  The denominator is again 6^4. There are $5^4 = 625$ ways to *not* roll any sixes, so there are $1296 - 625 = 671$ ways to throw at least one six. The probability is thus $671/1296$ which is about 52%.

- (c, 5) If we compute a score for your roll of four dice, by adding up four points for each 6, three points for each 5, two points for each 4, and one point for each 3, with no points for any other numbers, what is the expected value for your score?

  Each die has an expected value of $4(1/6) + 3(1/6) + 2(1/6) + 1(1/6) = 10/6$. The expected value of the whole hand is the sum of the expected values of the four cards, which is $40/6$.

- (d, 5) A **straight** is four rolls with consecutive ranks, meaning 1-2-3-4, 2-3-4-5, or 3-4-5-6, in any order. A **three-of-a-kind** is a roll where any three of the four numbers are the same, and the fourth number is different. (An example of a three-of-a-kind would be a roll of 3-3-5-3.) **Are you more likely to roll a straight, or a three-of-a-kind?** Justify your answer.

  There are again $6^4 = 1296$ possible rolls, and we need to compute the number of possible straights and the number of possible three-of-a-kind rolls. There are $4! = 24$ ways to get 1-2-3-4, 24 for 2-3-4-5, and 24 for 3-4-5-6, so there are $3 \cdot 24 = 72$ possible straights. To get a three-of-a-kind, we have six choices for the number that occurs three times, five choices for the number that occurs once, and four choices for in which position the different number occurs. There are thus $6 \cdot 5 \cdot 4 = 120$ possible three-of-a-kind rolls, more than the number of straights.

  (We could compare the probabilities as $72/1296$ and $120/1296$, but since we know that the two denominators are the same, we can compare the probabilities by comparing the numerators.)

  In poker dice with five dice, straights also rank above three of a kind – there we have 360 straights and 1200 three of a kind rolls.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Let D be a set and let $P(x)$ be a predicate defined on elements of D. If we know that the statement $\forall x \in D, P(x)$ is true, and y is any element of D, we cannot necessarily conclude that $P(y)$ is true.

  FALSE. The meaning of the statement $\forall x \in D, P(x)$ is that $P(y)$ is true for *every* possible y in D, so we can conclude that $P(y)$ is true.

  85% correct.

- (b) The set $\{\emptyset\}$ has no elements.

  FALSE. It has one element, which is $\emptyset$. The set $\emptyset$ has no elements but we said "$\{\emptyset\}$", which is the set containing the empty set and nothing else.

  33% correct.

- (c) If A is a finite set with m elements, and f is any function from A to A, then if we think of f as a relation, it contains exactly m pairs.

  TRUE. There is exactly one pair $(x, f(x))$ for every element x of A, no matter whether f is one-to-one.

  67% correct.

- (d) Let R be a symmetric relation on a finite set A, and let G be the directed graph for R. Then if we know that the edges $[x, y]$ and $[y, z]$ are both in G, we may conclude that the edge $[x, z]$ is also in G.

  FALSE. We would know this if R were transitive. But we only know that it is symmetric, not whether it is transitive.

  72% correct.

- (e) If A is a nonempty finite set and R is a binary relation on A that is a partial order, there must exist some element x in A such that there is no element y such that (x, y) is in R.

  FALSE. Since (x, x) is in R, because R is reflexive, and there is no condition that $x \neq y$, this cannot happen.

  54% correct.

- (f) Let A be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and let B be the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then the matrix product AB, using integer matrix multiplication, has all 0's.

  FALSE. The correct product has a 2 in the upper right. The matrix BA has all 0's.

  79% correct.

- (g) If you flip three fair coins, the probability that you get exactly two heads and one tails is exactly 3/8.

  TRUE. There are eight possible ways to flip three coins, and we are counting three of them: HHT, HTH, and THH. We could also compute $C(3, 2)(1/2)^2(1/2)^1 = 3/8$.

  69% correct.

- (h) If n is a positive integer, and k is an integer with $0 \leq k \leq n$, then the numbers $C(n, k)$ and $C(n, n-k)$ are equal.

  TRUE. We learned this identity. $C(n, k)$ is the number of ways to choose a set of k elements from an n-element set, and $C(n, n-k)$ is the number of ways to choose $n-k$ elements. But whenever you choose k elements, you are also choosing which are the $n-k$ elements that you are not picking.

  64% correct.

- (i) Players A and B are going to play two matches against one another. Let A_1 be the event of A winning the first match and A_2 the event of A winning the second match. If we know that $Prob(A_1) = 0.6$, $Prob(A_2|A_1) = 0.4$, and $Prob(A_2|\neg A_1) = 0.7$, then the probability that each player wins one of the matches is exactly 0.64.

  TRUE. The probability of $A_1 \cap A_2$ is $Prob(A_1)Prob(A_2|A_1) = 0.24$, the probability of $A_1 \cap \neg A_2$ is $Prob(A_1)(1 - Prob(A_2|A_1)) = 0.36$, the probability of $\neg A_1 \cap A_2$ is $(1 - Prob(A_1))Prob(A_2|\neg A_1) = 0.28$ The event of the two players splitting the matches is the sum of the second and third probabilities, $0.36 + 0.28 = 0.64$.

  62% correct.

- (j) Suppose you flip fair coins until the first time you get heads, after which you stop. Then the expected number of coins you flip before stopping is $(1/2) + (1/4) + (1/8) + (1/16) + \ldots = 1$.

  FALSE. We computed this expected value as 2. We could use a sum like this to get this answer, but what we want is $1(1/2) + 2(1/4) + 3(1/8) + \ldots = 2$.

  54% correct.