INFO 150
A Mathematical Foundation for Informatics
SOLUTIONS to Final Exam Fall 2025

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DIRECTIONS:

- Answer the problems on the exam pages.

- There are six problems on pages 2-7, some with multiple parts, for 120 total points. The scale is A = 105, B = 85, C = 65, D = 45, F = 25.

- If you need extra space use the back of a page.

- No books, notes, calculators, or collaboration.

- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer. But there may be times when you need to compare numerical answers to one another, which would mean computing them or at least estimating them.

1	/15
2	/20
3	/25
4	/20
5	/20
6	/20
Total	/120

Question 1 (15): Briefly identify and distinguish the following terms or concepts (3 points each):

- (a) the **inclusive or** and the **exclusive or** of two boolean variables p and q

  The value of the first, $p \vee q$, is true if both p and q are true, but the second, $p \oplus q$, is false in that case.

- (b) a **symmetric relation** on a set A and an **antisymmetric relation** on A

  For any two elements x and y of A, a symmetric relation has either both the pairs (x, y) and (y, x) or neither of them, while an antisymmetric relation can contain at most one of them, unless $x = y$.

- (c) a **set** and a **multiset** from some set S

  A set is a collection of items, such that every element is either in or out. A multiset can contain multiple copies of the same element.

- (d) an **event** and a **random variable**

  An event is a subset of an event space, which either occurs or doesn't, given any atomic event. A random variable is an assignment of a real number for each possible atomic event.

- (e) an **undirected graph** and a **directed graph** with the same set of nodes

  The edges of an undirected graph go between two distinct nodes, while an edge of a directed graph goes from some node to some node, possibly the same node.

Grading Note: *The mean score was 9.3/15, with 11% getting full marks.*

Question 2 (20): In a local election, a set of three school committee members will be chosen from the set $C = \{M, N, O, P, Q, R\}$ of six candidates.

Two (P and Q) are members of the Boring Party, two (M and R) are members of the Silly Party, and the other two are Independents. The order of candidates on the ballot will be chosen randomly, with each ordering being equally likely.

- (a, 5) How many possible ballot orders are there for the six candidates? (A typical order might be "O, M, Q, P, N, R.")

  There are $P(6,6) = 6! - 120$ possible orderings.

- (b, 5) How many of those candidates have a Boring candidate first, an Independent candidate third, and a Silly candidate last?

  There are two choices for the first Boring candidate, two choices for the third Independent candidate, and two for the sixth Silly candidate. So there are $2^3 = 8$ ways to pick those positions, leaving three other candidates for the other three positions, and since they could be in any of those three positions, there are $3! = 6$ ways to place them. The total number of orderings, given the Product Rule, is $8 \cdot 6 = 48$, and the probability that this happens, though we didn't ask, is $48/120 = 2/5 = 40\%$.

- (c, 5) What is the probability that M, a Silly candidate, comes before both of the Boring candidates? (**Hint:** There are at least two approaches to solve this. One is to consider the six cases on whether M comes in first, second, third, fourth, fifth, or sixth position. In each case, how many position come after M, and what is the chance that both Boring candidates are among those positions? The second method is to consider the *relative* positions of M and the two Boring candidates.)

  M is equally likely to have any of the six positions. If M is in position i, there are $6 - i$ positions after them, and the chance that the two Boring candidates are among those $6 - i$ positions is $C(6 - i, 2)/C(5, 2)$. So we have $(1/6)(10/10 + 6/10 + 3/10 + 1/10 + 0/10 + 0/10) = 1/3$. Another valid argument is to look at the relative position of M and the two Boring candidates. M is equally likely to be in the first, second, or third among those three, and in exactly one of those three choices does M come before the other two.

- (d, 5) In how many ways can the election of the three winners come out, if we view it in terms of the party affiliations of the winners? For example, one possible outcome would be "two Boring candidates and one Silly candidate were elected".

  There is one way in which one candidate from each party wins. Otherwise, two of one party win, and the other is from one of the other two parties. In this case we have three choices for which party gets two and then one choice of which other party gets the third slot. In all, we have 1 (BIS) plus six (BBI, BBS, BII, BSS, IIS, and ISS) with two of one party and one of another party.

Grading Note: *The mean score was 10.39/20, with one person getting full marks. For the most part, I gave at most 2/5 for wrong answers,*

Question 3 (25): These translations and proof deal with the scenario in Question 2. We define a function g from the set $C = \{M, N, O, P, Q, R\}$ to the set $Y = \{1, 2, 3, 4, 5, 6\}$, such that "$g(i) = j$" means "candidate i gets ballot position j". We also have a function h from C to the set $Z = \{B, I, S\}$ of the party affiliations, the values of which are given in Question 2.

- (a, 5) Write a symbolic statement meaning "Candidate M came after both Boring candidates, but before both Independent candidates."

  Solution: The easiest way to write this is as $(g(M) > g(P)) \wedge (g(M) > g(Q)) \wedge (g(M) < g(N)) \wedge (g(M) < g(O))$. But you could also use quantifiers and say, for example, $\forall X \in C : (h(X) = B \rightarrow g(M) > g(X) \wedge (h(X) = I \rightarrow g(M) < g(X))$.

- (b, 5) Write a symbolic statement meaning "Some Silly candidate was between the two Independent candidates".

  Solution: $\exists X \in C : (h(X) = S) \wedge \exists Y \in C : \exists Z \in C : (g(Y) < g(X)) \wedge (g(Z) > g(X))$.

- (c, 5) Translate into English: $\forall U \in C : \exists V \in C : (g(V) \leq g(U)) \wedge (h(V) = B)$

  Solution: "Given any candidate, there is a Boring candidate either before them or equal to them."

- (d, 10) Explain carefully why if the statements in part (a) and part (b) are both true, the statement in (c) must also be true. You may quote facts from the lecture and book if you do so clearly, and you may assume the function values for h given in Question 2.

  Solution: The second statement refers to M or R, and it can't be M because M is before the two Independents, not between them. So we know that R is between the two Independents, and in particular must be after M. Therefore M is exactly third in the order, after the two Bores and before the other three. The first candidate in the order, who may be either P or Q, can satisfy the given conditions for V in the third statement, since they are Boring and any candidate is either equal to them or comes after them.

Grading note: *The mean score was 15.82/25, with two people getting full marks.*

Question 4 (20): Here we define a function h from positive integers to integers, by the three rules $h(1) = 3$, $h(2) = 6$, $h(3) = 12$, and, if $n > 3$, $h(n) = 3h(n-2) + 2h(n-3)$. Our goal is to prove, by induction, for all positive integers n, that the function h is equal to $3 \cdot 2^{n-1}$.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

  Solution: $P(n)$ says that $h(n) = 3 \cdot 2^{n-1}$.

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

  Solution: We need to verify $P(1)$, $P(2)$, and $P(3)$, since the inductive rule is only useful for $P(n)$ with $n > 3$.
  Here $h(1)$ is defined to be 3, and $P(n)$ says that $h(1) = 3 \cdot 2^{1-1} = 3$.
  Here $h(2)$ is defined to be 6, and $P(n)$ says that $h(2) = 3 \cdot 2^{2-1} = 6$.
  Here $h(3)$ is defined to be 12, and $P(n)$ says that $h(3) = 3 \cdot 2^{3-1} = 12$.

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

  Solution: The IH says that for all i with $1 \leq m-1$, $P(i)$ is true. (In this case we will only need the statements $P(m-2)$ and $P(m-3)$.) The IG says that $P(m)$ is true, meaning that $h(m) = 3 \cdot 2^{m-1}$.

- (d, 8) Prove your inductive step, completing the proof.

  Solution: We need an inductive step for every m with $m \geq 4$. By the definition of the function, $h(m) = 3h(m-2) + 2h(m-3)$. By the IH applied to $m-2$ and $m-3$, this is $3(3 \cdot 2^{m-3}) + 2(3 \cdot 2^{m-4})$. We can rewrite this as $3 \cdot (3 \cdot 2^{m-3} + 2 \cdot 2^{m-4})$ which is $3 \cdot 2^{m-4}(3 \cdot 2 + 2) = 3 \cdot 2^{m-4} \cdot 8 = 3 \cdot 2^{m-1}$. So the function defined by the recurrence equals the claimed function for m, and $P(m)$ is true. This completes the induction and thus the proof.

Grading note: *The mean score was 11.93/20, with one student getting full marks.*

Question 5 (20): Suppose that we are given exactly two cards from a standard 52-card deck. We assume that each pair of cards is equally likely to be chosen. Let F ("flush") be the event that the two cards have the same suit (for example, if both are hearts). Let P ("pair") be the event that the two cards have the same rank (for example, if both are sixes).

- (a, 5) Compute the probabilities $\text{Prob}(F)$ and $\text{Prob}(P)$.

    Solution: There are $C(52, 2) = (52 \cdot 51)/2 = 1326$ possible pairs. The number in which F happens is $4 \cdot C(13, 2) = 4 \cdot 78 = 312$, because there are four suits, and 78 ways to choose two of the 13 cards in a given suit. So $\text{Prob}(F) = 312/1326 = 4/17$, about 0.235. We can also compute this value by considering the fact that if we pick one card, there are 12 of the remaining 51 cards that have the same suit as the first card.

    The number in which P happens is $13 \cdot C(4, 2) = 78$ because there are thirteen ranks, and six ways to choose two of the four cards of that rank. So $\text{Prob}(S) = 78/1326 = 1/17$. We can also compute this value by considering the fact that if we pick one card, there are 3 of the remaining 51 cards that have the same rank as the first card.

- (b, 5) Determine the probabilities $\text{Prob}(F \cap P)$ (the probability that both occur) and $\text{Prob}(F \cup P)$ (the probability that either or both occur).

    Solution: There is no way at all for the two cards to be both the same rank and the same suit, so $\text{Prob}(F \cap P) = 0$. Since the events F and P are disjoint, we can compute $\text{Prob}(F \cup P) = \text{Prob}(F) + \text{Prob}(P) = 4/17 + 1/17 = 5/17$.

- (c, 5) The **conditional probability** $\text{Prob}(F|P)$ is the fraction, out of all the cases where P occurs, that F also occurs. Compute $\text{Prob}(F|P)$. Then using Bayes' Theorem, or by any other valid means, also compute $\text{Prob}(P|F)$.

    Solution: Since none of the cases where P occur also have F occurring, $\text{Prob}(F|P) = 0$. We can also see, by the same reasoning, that $\text{Prob}(P|F)$ is also 0. To compute $\text{Prob}(P|F)$ using Bayes' Theorem, we would use the formula to get $\text{Prob}(P|F) = \text{Prob}(F|P)\frac{\text{Prob}(P)}{\text{Prob}(F)}$, getting $\text{Prob}(P|F) = 0 \cdot \frac{1/17}{4/17} = 0$.

- (d, 5) Suppose you pay \$10 to play a game in which you get two cards in this way, and you win \$20 if F happens, and \$80 if P happens. What is the expected value of this game?

    You pay the \$10 in all cases, you get \$20 with probability $4/17$, and you win \$80 with probability $1/17$. The expected value is $-10 + (4/17)(20) + (1/17)(50) = (1/17)(-170 + 80 + 80) = -10/17$, about -0.59 dollars.

Grading note: *The mean score was 10.79/20, with two students getting full marks.*

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

Grading note: *The mean score was 12.21/20, with the three best scores being 16/20. The individual scores are given below with the solutions.*

- (a) If Socrates is a man, and all men are mortal, then we may conclude that all men are Socrates.

  FALSE (100% correct). We should instead conclude that Socrates is mortal. (This comes up in Woody Allen's film *Love and Death***).**

- (b) Let p, q, and r be three boolean variables. If we are given that the compound propositions $p \rightarrow r$, $q \rightarrow r$, and $p \vee q$ are all true, then we may conclude that r is true.

  TRUE (86% correct). This is an example of the Proof By Cases rule.

- (c) We can prove the statement "$\forall x \in \mathbb{Z} : x^2 > 0$" to be false by finding a counterexample. **TRUE (61% correct). The counterexample is $x = 0$.**

- (d) It is not true that any given function is one-to-one if and only if it has exactly one output for each input in its domain.

  TRUE (21% correct). If two different inputs are mapped to the same output, it is not one-to-one, but could still be a function.

- (e) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then there are exactly six one-to-one functions from A to B.

  FALSE (61% correct). There are no one-to-one functions from A to B, since A is larger than B and any function must somewhere map two different elements to the same element of B. There *are* exactly six *onto* functions from A to B, since we get a function by choosing one of the two outputs for each input, there are $2^3 = 8$ total functions, and only two of them fail to be onto. This was graded incorrectly on the exam, and neither I nor any of the students who lost the points noticed.

- (f) Suppose I choose a five-letter string from the alphabet $\{a, b, c\}$, with every string being equally likely. Then the probability that I get one a, two b's, and two c's is exactly $30/243 = 10/81$.

  TRUE (71% correct). There are five positions for the a, and $C(4, 2)$ ways to chose two of the remaining four positions for the b's. This gives us 30 chances out of the $2^5 = 243$ total strings.

- (g) If I roll five six-sided dice as in Yahtzee^{TM} ("throwing 5D6"), the expected number of dice that show a prime number is exactly 2.5.

  TRUE (71% correct). There are three primes in the set $\{1, 2, 3, 4, 5, 6\}$, namely 2, 3, and 5, so the expected number of primes on each die is $3/6 = 1/2$, and we add the expected values for the five dice to get $5/2$.

- (h) Bayes' Theorem, which says that $\text{Prob}(A|B) = \text{Prob}(B|A)\frac{\text{Prob}(A)}{\text{Prob}(B)}$, is only valid when the events A and B are independent.

  FALSE (46% correct). It holds for any two random variables.

- (i) If an undirected graph has four nodes and four edges, then it must be connected.

  TRUE (39% correct). If a four-node graph is not connected, it must either have an isolated node (which would leave it at most three edges) or can be split into two two-node sets, with no edges from one set to the other set (which would leave it at most two edges).

- (j) Let G be a directed graph with a finite number n set of nodes, one loop on every node, and no other edges at all. Then the adjacency matrix of G would be the $n \times n$ identity matrix, with ones on its diagonal and zeros everywhere else.

  TRUE (75% correct). The ones would represent the single loop for each node, and the zeros would represent the lack of any edges from any node to any other node.