NAME: ____________________

SPIRE ID: ____________________

INFO 150
A Mathematical Foundation for Informatics
Final Exam Fall 2025

D. A. M. Barrington

16 December 2025

DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 120 total points. Probable scale is somewhere around A=105, C=70, but will be determined after I grade the exam.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer. But there may be times when you need to compare numerical answers to one another, which would mean computing them or at least estimating them.

1	/15
2	/20
3	/25
4	/20
5	/20
6	/20
Total	/120

Question 1 (15): Briefly identify and distinguish the following terms or concepts (3 points each):

- (a) the **inclusive or** and the **exclusive or** of two boolean variables p and q

- (b) a **symmetric relation** on a set A and an **antisymmetric relation** on A

- (c) a **set** and a **multiset** from some set S

- (d) an **event** and a **random variable**

- (e) an **undirected graph** and a **directed graph** with the same set of nodes

Question 2 (20): In a local election, a set of three school committee members will be chosen from the set $C = \{M, N, O, P, Q, R\}$ of six candidates.

Two (P and Q) are members of the Boring Party, two (M and R) are members of the Silly Party, and the other two are Independents. The order of candidates on the ballot will be chosen randomly, with each ordering being equally likely.

- (a, 5) How many possible ballot orders are there for the six candidates? (A typical order might be "O, M, Q, P, N, R.")

- (b, 5) How many of those candidates have a Boring candidate first, an Independent candidate third, and a Silly candidate last?

- (c, 5) What is the probability that M, a Silly candidate, comes before both of the Boring candidates? (**Hint:** There are at least two approaches to solve this. One is to consider the six cases on whether M comes in first, second, third, fourth, fifth, or sixth position. In each case, how many position come after M, and what is the chance that both Boring candidates are among those positions? The second method is to consider the *relative* positions of M and the two Boring candidates.)

- (d, 5) In how many ways can the election of the three winners come out, if we view it in terms of the party affiliations of the winners? For example, one possible outcome would be "two Boring candidates and one Silly candidate were elected".

Question 3 (25): These translations and proof deal with the scenario in Question 2. We define a function g from the set $C = \{M, N, O, P, Q, R\}$ to the set $Y = \{1, 2, 3, 4, 5, 6\}$, such that "$g(i) = j$" means "candidate i gets ballot position j". We also have a function h from C to the set $Z = \{B, I, S\}$ of the party affiliations, the values of which are given in Question 2.

- (a, 5) Write a symbolic statement meaning "Candidate M came after both Boring candidates, but before both Independent candidates."

- (b, 5) Write a symbolic statement meaning "Some Silly candidate was between the two Independent candidates".

- (c, 5) Translate into English: $\forall U \in C : \exists V \in C : (g(V) \leq g(U)) \wedge (h(V) = B)$

- (d, 10) Explain carefully why if the statements in part (a) and part (b) are both true, the statement in (c) must also be true. You may quote facts from the lecture and book if you do so clearly, and you may assume the function values for h given in Question 2.

Question 4 (20): Here we define a function h from positive integers to integers, by the three rules $h(1) = 3$, $h(2) = 6$, $h(3) = 12$, and, if $n > 3$, $h(n) = 3h(n-2) + 2h(n-3)$. Our goal is to prove, by induction, for all positive integers n, that the function h is equal to $3 \cdot 2^{n-1}$.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

- (d, 8) Prove your inductive step, completing the proof.

Question 5 (20): Suppose that we are given exactly two cards from a standard 52-card deck. We assume that each pair of cards is equally likely to be chosen. Let F ("flush") be the event that the two cards have the same suit (for example, if both are hearts). Let P ("pair") be the event that the two cards have the same rank (for example, if both are sixes).

- (a, 5) Compute the probabilities $\text{Prob}(F)$ and $\text{Prob}(P)$.

- (b, 5) Determine the probabilities $\text{Prob}(F \cap P)$ (the probability that both occur) and $\text{Prob}(F \cup P)$ (the probability that either or both occur).

- (c, 5) The **conditional probability** $\text{Prob}(F|P)$ is the fraction, out of all the cases where P occurs, that F also occurs. Compute $\text{Prob}(F|P)$. Then using Bayes' Theorem, or by any other valid means, also compute $\text{Prob}(P|F)$.

- (d, 5) Suppose you pay \$10 to play a game in which you get two cards in this way, and you win \$20 if F happens, and \$80 if P happens. What is the expected value of this game?

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) If Socrates is a man, and all men are mortal, then we may conclude that all men are Socrates.

- (b) Let p, q, and r be three boolean variables. If we are given that the compound propositions $p \rightarrow r$, $q \rightarrow r$, and $p \vee q$ are all true, then we may conclude that r is true.

- (c) We can prove the statement "$\forall x \in \mathbb{Z} : x^2 > 0$" to be false by finding a counterexample.

- (d) It is not true that any given function is one-to-one if and only if it has exactly one output for each input in its domain.

- (e) Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then there are exactly six one-to-one functions from A to B.

- (f) Suppose I choose a five-letter string from the alphabet $\{a, b, c\}$, with every string being equally likely. Then the probability that I get one a, two b's, and two c's is exactly $30/243 = 10/81$.

- (g) If I roll five six-sided dice as in Yahtzee^{TM} ("throwing 5D6"), the expected number of dice that show a prime number is exactly 2.5.

- (h) Bayes' Theorem, which says that $\text{Prob}(A|B) = \text{Prob}(B|A)\frac{\text{Prob}(A)}{\text{Prob}(B)}$, is only valid when the events A and B are independent.

- (i) If an undirected graph has four nodes and four edges, then it must be connected.

- (j) Let G be a directed graph with a finite number n set of nodes, one loop on every node, and no other edges at all. Then the adjacency matrix of G would be the $n \times n$ identity matrix, with ones on its diagonal and zeros everywhere else.