NAME: ______________________________

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INFO 150
A Mathematical Foundation for Informatics
Final Exam Fall 2024

D. A. M. Barrington

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 120 total points. The scale was A = 105, B = 87, C = 69, D = 51, F = 33.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer. But there may be times when you need to compare numerical answers to one another, which would mean computing them or at least estimating them.

1	/15
2	/20
3	/25
4	/20
5	/20
6	/20
Total	/120

Question 1 (15): Briefly identify and distinguish the following terms or concepts (3 points each):

- (a) a **tautology** and a **contradiction** in boolean logic

- (b) an **onto function** and a **one-to-one function**

- (c) the **probability** of an event and the **number of ways** an event could occur

- (d) a **directed graph** and an **undirected graph** in graph theory

- (e) a **walk** and a **path** in an undirected graph

Question 2 (20): Seven women from New England state universities are about to compete in a cross-country footrace. They will finish in order from 1 through 7 – there are no ties. The competitors, with their states, are Abigail (MA), Brittany (CT), Carmen (RI), Debra (MA), Enid (MA), Francine (CT), and Gretchen (VT). We define a function f from the set $R = \{A, B, C, D, E, F, G\}$ to the set $P = \{1, 2, 3, 4, 5, 6, 7\}$, with the rule being that $f(X) = n$ if competitor X finishes in position n.

- (a, 5) In how many ways could the race come out? (For example, one of the ways you are counting is "$f(A) = 3$, $f(B) = 7$, $f(C) = 2$, $f(D) = 4$, $f(E) = 1$, $f(F) = 6$, and $f(G) = 5$".

- (b, 5) In how many of those possible outcomes does Abigail finish third, *and* Enid finish either fourth or fifth, *and* Francine finish first?

- (c, 5) If each possible order of finish were equally likely, what is the the probability that the three Massachusetts competitors would be the top three finishers? (For example, this would happen if the order were "D, A, E, G, F, B, C".)

- (d, 5) Let's consider a "team result" for this race where we list the *states* of each finisher rather than their names. (For example, the team result for the race result example in part (c) would be "MA, MA, MA, VT, CT, CT, RI". How many different team results are possible for this race?

Question 3 (25): These translations and proof deal with the scenario in Question 2. Remember that if X finishes *before* Y, then $f(X) < f(Y)$.

- (a, 5) Write a symbolic statement meaning "Enid placed fourth, and Gretchen finished before both Debra and Francine."

- (b, 5) Write a symbolic statement meaning "Some person finished after Enid but before both Debra and Francine."

- (c, 5) Translate into English: $((f(A) = 1) \wedge (f(C) = 3) \wedge (f(G) < f(B))) \rightarrow (f(G) = 2)$.

- (d, 10) Explain carefully why if the statements in part (a) and part (b) are both true, the statement in (c) must also be true. You may quote facts from the lecture and book if you do so clearly.

Question 4 (20): In this problem we define a function G from positive integers to positive integers, defined recursively. It uses the rules $G(1) = 1$, $G(2) = 4$, $G(3) = 9$, and for all n with $n \geq 4$, $G(n) = G(n-1) - G(n-2) + G(n-3) + 4n - 6$. Here we ask you to prove, by induction on n, that for every positive natural n, $G(n)$ is equal to n^2.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

- (d, 8) Prove your inductive step, completing the proof.

Question 5 (20): In a particular dice game, you roll four fair six-sided dice, so that each number comes from the set $\{1, 2, 3, 4, 5, 6\}$ with equal probability.

- (a, 5) What is the probability that the four numbers you rolled are all different?
- (b, 5) What is the probability that you rolled at least one six?
- (c, 5) If we compute a score for your roll of four dice, by adding up four points for each 6, three points for each 5, two points for each 4, and one point for each 3, with no points for any other numbers, what is the expected value for your score?
- (d, 5) A **straight** is four rolls with consecutive ranks, meaning 1-2-3-4, 2-3-4-5, or 3-4-5-6, in any order. A **three-of-a-kind** is a roll where any three of the four numbers are the same, and the fourth number is different. (An example of a three-of-a-kind would be a roll of 3-3-5-3.) **Are you more likely to roll a straight, or a three-of-a-kind?** Justify your answer.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Let D be a set and let $P(x)$ be a predicate defined on elements of D. If we know that the statement $\forall x \in D, P(x)$ is true, and y is any element of D, we cannot necessarily conclude that $P(y)$ is true.

- (b) The set $\{\emptyset\}$ has no elements.

- (c) If A is a finite set with m elements, and f is any function from A to A, then if we think of f as a relation, it contains exactly m pairs.

- (d) Let R be a symmetric relation on a finite set A, and let G be the directed graph for R. Then if we know that the edges $[x, y]$ and $[y, z]$ are both in G, we may conclude that the edge $[x, z]$ is also in G.

- (e) If A is a nonempty finite set and R is a binary relation on A that is a partial order, there must exist some element x in A such that there is no element y such that (x, y) is in R.

- (f) Let A be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, and let B be the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. Then the matrix product AB, using integer matrix multiplication, has all 0's.

- (g) If you flip three fair coins, the probability that you get exactly two heads and one tails is exactly 3/8.

- (h) If n is a positive integer, and k is an integer with $0 \leq k \leq n$, then the numbers $C(n, k)$ and $C(n, n-k)$ are equal.

- (i) Players A and B are going to play two matches against one another. Let A_1 be the event of A winning the first match and A_2 the event of A winning the second match. If we know that $Prob(A_1) = 0.6$, $Prob(A_2|A_1) = 0.4$, and $Prob(A_2|\neg A_1) = 0.7$, then the probability that each player wins one of the matches is exactly 0.64.

- (j) Suppose you flip fair coins until the first time you get heads, after which you stop. Then the expected number of coins you flip before stopping is $(1/2) + (1/4) + (1/8) + (1/16) + \ldots = 1$.