NAME: ______________________________

SPIRE ID: ______________________________

INFO 150
A Mathematical Foundation for Informatics
SOLUTIONS to Second Midterm

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 100 total points. The scale will be A = 88, B = 72, C = 56, D = 40, F = 24.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/15
3	/20
4	/15
5	/15
6	/20
Total	/100

Question 1 (15): Briefly explain the meaning of each of these terms or concepts (3 points each):

- (a) the **power set** of a set
  The power set of a set S is the set of all subsets of S, that is, $\{T : T \subseteq S\}$.

- (b) a **symmetric** binary relation
  A binary relation R is symmetric if whenever a pair (x, y) is in R, the pair (y, x) is also in R.

- (c) the **Inclusion/Exclusion Principle** for two sets, also called the **Double Counting Rule**
  If A and B are any two finite sets, the size $|A \cup B|$ of the set $A \cup B$ is $|A| + |B| - |A \cap B|$.

- (d) the **codomain** of a function
  The codomain of a function $f : A \to B$ is the B, the set of possible outputs for the function.

- (e) when a binary relation on a set is a **partial order**
  A binary relation is a partial order if it is reflexive, antisymmetric, and transitive.

Question 2 (15): For any positive integer n, let $S(n)$ be the sum $\sum_{i=1}^{n} i^3$. That is, $S(n)$ is the sum of the numbers $1^3 + 2^3 + \ldots + n^3$. For example, $S(1) = 1$ and $S(3) = 1 + 8 + 27 = 36$. We would like to prove by induction that for all positive naturals n, $S(n)$ is given by the closed formula $S(n) = (\frac{n^2+n}{2})^2$.

- (a, 3) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.
  $P(n)$ **says that** $S(n) = (\frac{n^2+n}{2})^2$**.**

- (b, 3) State and prove the **base case** (or base cases) for your induction.
  We only need one base case, for $P(1)$**. It says that** $S(1) = (\frac{1^2+1}{2})^2$**. The right-hand side evaluates to** $(\frac{1+1}{2})^2 = 1$**, and we know that** $S(1) = 1$ **is given.**

- (c, 3) State the **inductive hypothesis** and **inductive goal** for your inductive step.
  We only need the statement $P(m-1)$**, though we could include the other statements** $P(1)$ **through** $P(m-2)$ **if we like, and** $P(m-1)$ **says that** $S(m-1) = (\frac{(m-1)^2+(m-1)}{2})^2$**. The IG says that** $P(m)$ **is true, which says that** $S(m) = (\frac{m^2+m}{2})^2$**.**

- (d, 6) Prove your inductive step, completing the proof.
  We know that $S(m) = S(m-1) + m^3$**. Applying the IH,** $S(m-1) = (\frac{(m-1)^2+(m-1)}{2})^2$**. We can rewrite** $S(m-1)$ **as** $(\frac{(m-1)m}{2})^2 = \frac{(m-1)^2m^2}{4}$**. Adding this term to** $m^3 = \frac{4m^3}{4}$**, we get** $\frac{m^2(m^2-2m+1+4m)}{4}$**, which is** $(\frac{m(m+1)}{2})^2$**, proving that** $P(m)$ **is true.**

Question 3 (20): Here we define a sequence of positive integers by the rules $G(1) = 1$, $G(2) = 3$, $G(3) = 9$, and for all n with $n \geq 4$, $G(n) = 8G(n-2) + 3G(n-3)$. Prove by induction, for all positive integers n with $n > 1$, that $G(n) = 3^{n-1}$.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

  $P(n)$ **says that** $G(n) = 3^{n-1}$**.**

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

  We need three base cases, for $P(1)$, $P(2)$, **and** $P(3)$, **because the last rule can only be applied if** $n \geq 4$. **All three cases are true because** $G(1) = 1 = 3^{1-1}$, $G(2) = 3 = 3^{2-1}$, **and** $G(3) = 9 = 3^{3-1}$**.**

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

  The IH says that for all i **with** $1 \leq m-1$, $P(i)$ **is true. The IG says that** $P(m)$ **is true, meaning that** $G(m) = 3^{m-1}$**.**

- (d, 8) Prove your inductive step, completing the proof.

  We need an inductive step for every m **with** $m \geq 4$**. By the rule,** $G(m) = 8G(m-2)+3G(m-3)$**. By the IH applied to** $m-2$ **and** $m-3$, $G(m-2) = 3^{m-3}$ **and** $G(m-3) = 3^{m-4}$**. Thus** $G(m)$ **evaluates to** $8 \cdot 3^{m-3} + 3 \cdot 3^{m-4}$**. Since** $3 \cdot 3^{m-4} = 3^{m-3}$, $G(m)$ **evaluates to** $9 \cdot 3^{m-3} = 3^{m-1}$**, satisfying the statement** $P(m)$**.**

Question 4 (15): Let A be the set $\{a, b, c, d\}$, let B be the set $\{1, 2, 3, 4\}$. Let $R \subseteq A \times B$ be the relation $\{(a, 3), (b, 1), (d, 2)\}$, $S \subseteq A \times B$ be the relation $\{(a, 2), (c, 4), (d, 2)\}$, and $T \subseteq A \times B$ be the relation $\{(b, 4), (c, 4), (d, 2)\}$.

- (a, 5) Which of the relations $R \cup S$, $R \cup T$, $S \cup T$, and $R \cup S \cup T$, if any, are functions from A to B? Explain your answers.

  $R \cup S$ **is not because it maps** a **to both** 2 **and** 3**.** $R \cup T$ **maps** b **to both** 1 **and** 4**.** $S \cup T$ **is because it maps** a **only to** 2**,** b **only to** 4**,** c **only to** 4**, and** d **only to** 2**.** $R \cup S \cup T$ **is not because it has the same violations as does** $R \cup S$ **and** $R \cup T$**.**

- (b, 5) Explain why $R \cup (S \cap T)$ is a function from A to B.

  Each element of A **is mapped to exactly one element of** B**:** a **to** 3**,** b **to** 1**,** c **to** 4**, and** d **to** 2**.**

- (c, 5) Is $R \cup (S \cap T)$ an invertible function? If so, describe its inverse and explain why it is the inverse. If not, justify your claim that it is not.

  This function is invertible, because its inverse $\{(1, b), (2, d), (3, a), (4, c)\}$ **maps each element of** B **to exactly one element of** A**.**

Question 5 (15): Let $f : A \to B$ and $g : B \to C$ be any functions, and let $h : A \to C$ be the **composition** of f and g, which is the function whose rule is "$h(x) = g(f(x))$".

Recall the definition of an **onto function**: $f : A \to B$ is one-to-one if $\forall y \in B, \exists x \in A, f(x) = y$.

1. (a, 10) Explain why, if f and g are both onto functions, h is also a onto function.

   The statement that h is onto is $\forall z \in C, \exists x \in A, h(x) = z$. We need to show if the Reader chooses any alleged counterexample to the statement "$exists x \in A, h(x) = z$" must fail. Suppose Reader chooses an element z. Because g is onto, we know that there exists an element y of B such that $g(y) = z$. Because f is onto, there must exist an element x of A such that $f(x) = y$. But then $h(x) = g(f(x) = g(y) = z$, and z is not a counterexample.

2. (b, 5) If h is a onto function, is it necessarily true that f and g are both onto functions? Explain your answer.

   It is not necessarily true. Let A have one element x, B have two elements y and y', and C have one element z. Let $f(x) = y$ and $g(y) = z$, so that $h(x) = z$ and h is onto (since every element of C has an element of A that is mapped to it by h). But in this case f is not onto, because the element y' of B is not mapped to any element of A by f.

   Many of the proposed counterexamples were slightly wrong because their g was not a function, since they didn't assign any value to the element in B that was not hit by f. Any value would do, but there has to be one.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Define a sequence of real numbers by the rule $a_1 = 2$, $a_2 = 1$ and, for all n with $n > 2$, $a_n = (a_{n-1} + a_{n-2})/2$. Then for all positive integers n, a_n is given by the closed form $a_n = (4/3) + (2/3)(-1/2)^n$.

  FALSE. The correct closed form is $(4/3) + (2/3)(-1/2)^{n-1}$. You could prove the inductive case of this, but the base case is wrong.

- (b) If the preconditions of a program are true, and it terminates, then its postconditions will be true.

  FALSE. This is true if the program is partially correct, but we did not say that it is.

- (c) Let $P(n)$ be a predicate over the positive integers. If we prove $(P(m-2) \wedge P(m-3)) \rightarrow P(m)$ for all m with $m \leq 4$, and we prove $P(1)$, $P(2)$, and $P(3)$, then it may still be possible that for some positive integer n, $P(n)$ is false.

  TRUE. These steps *would* form a valid proof by induction that $P(n)$ is true for all positive integers n, except that I mistakenly wrong "$m \leq 4$" rather than "$m \geq 4$", making my inductive step completely useless. A student pointed me to the error late in the in-person exam. This made the true/false question too tricky, in my opinion, for the level of this course. So I am giving credit for both "FALSE" and "TRUE" answers on this question.

- (d) Let $Q(n)$ be a predicate over the positive integers, such that $Q(3)$ is false, and such that for all n such that $n \geq 9$, $Q(n) \rightarrow Q(n-8)$ and $Q(n-8) \rightarrow Q(n)$ are both true. Then it is not possible that for some positive integer k, $Q(k)$, $Q(k+1)$, $Q(k+2)$, $Q(k+2)$, $Q(k+3)$, $Q(k+4)$, $Q(k+5)$, and $Q(k+6)$ are all true.

  FALSE. There cannot be *eight* consecutive true values, because one of them would be of the form $Q(8m+3)$, and we can prove by induction that $Q(8m+3)$ is false for all m. But there's nothing to stop *seven* consecutive values to all be true.

- (e) The statement $A \cup (B \cap C') = (A \cap B) \cup (A \cap C')$ is a set identity.

  FALSE. This is a garbled version of the Distributive Law. If B and C' are disjoint, the left-hand set is empty, but if any element is in both A and B, the right-hand side is not.

- (f) If A and $A \cup B$ are both infinite sets, then it is possible that B is finite, and it is also possible that B is infinite.

  TRUE. The union of an infinite set and another set is infinite, whether the second is finite or infinite.

- (g) If $R \subseteq A \times A$ and $S \subseteq A \times A$ are two symmetric relations on the same set, then $R \cap S$ is a symmetric relation but $R \cup S$ need not be symmetric.

  FALSE. Both $R \cap S$ and $R \cup S$ must be symmetric. For example, if the pair (x, y) is in $R \cup S$, it must be in either R or in S, and either condition forces (y, x) to be in the same set, putting it into $R \cup S$.

- (h) If a and b are any two real numbers with $a \neq 0$, then the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with the rule $f(x) = ax + b$ is invertible.

  Its inverse is $g(y) = (y - b)/a$.

- (i) Let $f : A \to B$ be a function where A is an infinite set and B is a finite set. Then f must be an onto function. (**Note:** The term "onto function" is defined in Question 5.)

  FALSE. If B has more than one element, and f maps every element of A into the same element of B, f is not onto.

- (j) A partial order R must contain at least two elements x and y such that $R(x, y)$ and $R(y, x)$ are both false.

  FALSE. That would be true if R were a partial order that is not a total order, but total orders are also partial orders.