NAME: ______________________________

SPIRE ID: ______________________________

INFO 150
A Mathematical Foundation for Informatics
Solutions to Second Midterm Exam Fall 2025

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 100 total points. Final scale was A =85, B = 70, C = 55, D = 40, F = 25.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/15
3	/15
4	/15
5	/20
6	/20
Total	/100

Question 1 (15): Briefly identify and distinguish the meaning of each of these terms or concepts (3 points each):

- (a) the **inductive hypothesis** of an induction and its **inductive goal**

  The IH is the statement being assumed, $P(1)$, $P(2)$,..., $P(m-1)$.
  The IG is the statement $P(m)$ that you are trying to prove in the inductive step.

  I accepted some inductive goals of $P(n+1)$, but the corresponding IH should still have $P(1)$ through $P(n)$ to match the book's definition.

- (b) the **intersection** of two sets A and B and the **union** of those two sets

  If A and B are any two sets, the intersection is the set of elements that are each both in A and in B. The union is the set of elements that are in either A or B, including any elements that are in both.

  These were mostly correct.

- (c) an **antisymmetric** binary relation and a binary relation that is **not symmetric**

  An antisymmetric binary relation R is one where there are never two different elements x and y such that (x, y) and (y, x) are both in R. A non-symmetric binary relation is one in which there *are* one or more elements x and y such that (x, y) is in R but (y, x) is not.

  A lot of you thought they were the same, and many were unclear on the two definitions.

- (d) a **function** from X to Y and an **onto** function from X to Y

  A function from X to Y is a binary relation where for every x in X, there is exactly one y in Y such that (x, y) is in the relation. An onto function from X to Y is a binary relation where (in addition to the property for being a function), for every y in Y, there is at least one x in X such that (x, y) is in the relation.

  I wanted both definitions.

- (e) the **rule of sums** and the **rule of sums with overlap** in counting two finite sets A and B

  If A and B are any two finite sets, the rule of sums says that *if* $A \cap B = \emptyset$, $n(A \cup B) = n(A) + n(B)$. The rule of sums with overlap says that for any finite sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

  A key point here was when *the ordinary Sum Rule applies.*

The average score for Question 1 was 9.8/15.

Question 2 (15): Here we define a function f from positive integers to positive integers, using the following recursive definition. We define $f(1)$ to be 1. If n is even, then $f(n)$ is defined to be $f(n/2)$. If n is odd and $n > 1$, then $f(n)$ is defined to be $f(n-2)+2$. Your goal is to prove, for all positive integers n, that $f(n)$ is odd.

- (a, 3) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

  $P(n)$ **says that** $f(n)$ **is odd.**

  The most common error was to define $P(n)$ as "$f(n)$ is odd for all positive integers", which is wrong because $P(n)$ refers only to the number n, not to all numbers.

- (b, 3) State and prove the **base case** (or base cases) for your induction.

  The base case is $P(1)$**, which says that** $f(1)$ **is odd, and this is true because** $f(1) = 1$ **and** 1 **is odd.**

  It was fine to do more base cases, but if you left out $P(1)$ you were wrong.

- (c, 3) State the **inductive hypothesis** and **inductive goal** for your inductive step.

  The IH says that $P(i)$ **is true for all** i **such that** $1 \leq i \leq m-1$**, so that for each such** i**,** $f(i)$ **is odd.**

  The IG says that $P(m)$ **is true, so that** $f(m)$ **is odd.**

  It is insufficient for your IH to only include $P(m-1)$, because you will need $P(m-2)$ and $P(m/2)$.

- (d, 6) Prove your inductive step, completing the proof.

  We need two cases, depending on whether m **is even or odd.**

  If m **is even,** $m/2$ **is in the range from** 1 **through** $m-1$**, so the IH tells us that** $f(m/2)$ **is odd. Since** $f(m)$ **equals this odd number, it is itself odd and the IG is true.**

  If m **is odd and** $m > 1$**,** $m-2$ **is in the range from** 1 **through** $m-1$**, so the IH tells us that** $P(m-2)$ **is true, and thus that** $f(m-2)$ **is odd. Therefore** $f(m) = 2 + f(m-2)$ **is odd, as the sum of an even number and an odd number.**

  Many of you only covered one of the cases, but both are needed.

The average score on Question 2 was 8/9/15.

Question 3 (15): Let D be a set of dogs, including a set L of Labradors, a set R of retrievers, and a set T of terriers.

- (a, 3) We are told that "Every Labrador is a retriever, but no retriever is a terrier." Express this statement using subsets, using set operations and the relations $\subseteq$ and/or $=$ on the given sets.

  $L \subseteq R$ and $R \cap T = \emptyset$ (there are other possibilities such as $T \subseteq R'$ or $R \subseteq T'$for the second fact).

  A common answer was $\neg(R \subseteq T)$, but that is wrong. To say "R is not a subset of T" means "there exists a retriever that is not a terrier", which is quite different from "no retriever is a terrier".

- (b, 3) Draw a Venn diagram showing the relationship of all four sets D, L, R, and T as given at the beginning of the question and in part (a).

  We have a box for the universal set D. Within it we have a loop for R, with another loop completely within it for L, and a separate loop for T that is separate from R.

- (c, 3) Here is another statement: "Every terrier is not a Labrador". What would be a *counterexample* for this statement?

  A counterexample would be a dog that is both a terrier and a Labrador.

- (d, 6) Write a letter to the Reader that should convince them, given the statement in part (a), that they should not look for counterexamples to the statement in part (c).

  "Dear Reader,

  You are looking for a dog that is both a terrier and a Labrador. Suppose you found a dog x with those properties. Since x is a Labrador, the first part of the statement means that x is also a retriever. But this would contradict the second statement, because it says that no retriever is a terrier, but x would be both a retriever and a terrier."

  It's important that you are arguing on the basis of the given statements, rather than from inferences you have made in the earlier parts of the problem.

The average score on Question 3 was 11.9/15.

Question 4 (15): Consider two functions from $\mathbb{Z}$ (the integers, including negative numbers) to $\mathbb{Z}$. For any integer n, g is defined such that $g(n) = 2n-3$ and h is defined such that $h(n) = n-4$.

- (a, 3) What functions are the **compositions** $g \circ h$ and $h \circ g$?

  The function $g \circ h$ is defined such that $(g \circ h)(n) = g(h(n)) = 2(n-4)-3 = 2n-11$.
  The function $h \circ g$ is defined such that $(h \circ g)(n) = h(g(n)) = (2n-3)-4 = 2n-7$.

- (b, 4) Are either or both of the functions g and h **one-to-one functions** (injections)? Justify your answers.

  Both are one-to-one. We cannot have different numbers x and y such that either $g(x) = g(y)$ or $h(x) = h(y)$. If $g(x) = g(y)$, the definition would tell us that $2x - 3 = 2y - 3$, and we can prove $x = y$ by algebra. Similarly, if $h(x) = h(y)$, we would have $x - 4 = y - 4$, from which we can prove $x = y$ by adding 4 to both sides of the equation.

  Lots of people got the definition of "one-to-one" wrong (many switched it with the definition for "onto".

  In these next three problems, I divided the four points into one for each of the two boolean questions "does g have the property" and "does h have the property", deducted another point if your reasoning was bad, and deducted another point if you gave no reasoning at all.

- (c, 4) Are either of both of the functions g and h **onto functions** (surjections)? Justify your answers.

  The function g is not onto. For any integer n, $2n$ is an even number, and so $2n - 3$ is an odd number. There can be no n such that $g(n)$ is even, so it is not true that every integer occurs as $g(n)$ for some n.

  The function h, however, is onto. For any integer m, $m = h(m + 4)$, and $m + 4$ is also an integer, so there exists an integer n such that $h(n) = m$.

- (d, 4) Do either or both of the functions g and h have **inverse functions**? Give the inverses, if any, where they exist.

  The function g does not have an inverse, because for any even number m, such as 0, there does not exist any n such that $g(n) = m$. But h does have an inverse, because we can define the function k so that $k(m) = m + 4$. Then $k \circ h$ and $h \circ k$ are each the identity function, that is, $h(k(m)) = m$ and $k(h(n)) = n$, so k is the inverse function of h.

The average score on Question 4 was 9.6/15.

Question 5 (20): In this problem, we consider randomly choosing a three-letter word, where each letter is chosen from the set $\{A, B, C, D, E\}$. We assume that in each of the three positions, each of the five letters are equally likely to be chosen. We also assume that the choices of the three letters are **independent**.

1. (a, 3) In how many ways could the three letters be chosen?
   Here we are counting sequences, n^r **where** $n = 5$ **and** $r = 3$**, so** $5^3 = 125$**.**

2. (b, 2) What is the probability that the word chosen is exactly "EAD"?

   The probability is $1/125 = 0.008$**.**
3. (c, 3) In how many different ways could the word be chosen with three different letters? (For example, "EAD" would be one of these words, but "EAE" would not be.)

   Here we are counting no-repeat sequences, $P(5, 3) = 5 \times 4 \times 3 = 60$**.**
4. (d, 2) What is the probability that the event of part (c) happens?

   The probability is $60/125 = 0.48$**.**
5. (e, 3) In how many ways could the word be chosen such that the letters are all different, *and* that the letters occur in order? (For example, "ADE" would be one of these words, but "AED" would not be.)

   Now we are counting sets, $C(5, 3) = P(5, 3)/3! = (5 \times 4 \times 3)/(1 \times 2 \times 3) = 60/6 = 10$**.**
6. (f, 2) What is the probability that the event of part (e) happens?

   The probability is $10/125 = 0.08$**.**
7. (g, 3) In how many ways could the word be chosen such the letters come in order, whether or not the letters are different? (For example, "AAD", "BCC", "EEE" and "ABE" would all be among these strings, but "CBC" or "BBA" would not be.)
   Now we are counting multisets with $n = 5$ **and** $r = 3$**, where we can use the stars-and-bars construction with three 0's and four 1's, so the number of ways to do it is** $C(7, 3) = C(7, 4) = (7 \times 6 \times 5)/(1 \times 2 \times 3) = 35$**.**

8. (h, 2) What is the probability that the event of part (g) happens?
   The probability is $35/125 = 0.28$**.**

The average score for Question 5 was 16.0/20.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Let $P(n)$ be a predicate on the positive integers. If $P(1)$ is true, and for all n with $n > 2$ we know that $P(n-2) \rightarrow P(n)$, we *cannot* be confident that $P(n)$ is true for all positive integers.
  TRUE (48% correct). There is no reason to believe that $P(2)$ is true.

- (b) If $P(m-1) \rightarrow P(m)$ is true for all positive integers m, then we know that $P(n)$ is true for all positive integers n.
  FALSE (36% correct). We are not given the base case. If $P(n)$ were *never* true, this implication would still be true vacuously.

- (c) Let $C = \{4k+1 : k \in \mathbb{Z}\}$ and $D = \{4k+3 : k \in \mathbb{Z}\}$. Whatever sets A and B are, we can be sure that $\{A, B, C, D\}$ is not a partition of $\mathbb{Z}$.
  TRUE (44% correct). C and D each contain all the odd numbers, so they are not disjoint.

- (d) The set $\{m^2 : m \in \mathbb{Z} \wedge (-4 \leq m \leq 4)93033\}$ has fewer than nine elements.
  TRUE (36% correct). It has five: 0, 1, 4, 9, and 16.

- (e) If a binary relation on a nonempty set A has no pairs in it at all, it is symmetric and transitive, but is not reflexive.
  TRUE (48% correct). The symmetric and transitive properties require some pairs to be there *if* other pairs are there, and are satisfied vacuously if there are no pairs at all. But reflexivity requires that (x, x) must be in the relation, and since A is non-empty, there is an element x in A, and thus (x, x) would have to be there.

- (f) Let A and B be any two nonempty sets, and let R be a relation from A to B. If there is an element $x \in A$ such that there is no pair (x, y) in R for any y in B, then R is not a function.
  TRUE (72% correct). Part of the definition of a function is that every x has at least one y.

- (g) If A, B, and C are any three finite sets, the size of $A \cup B \cup C$ (called $|A \cup B \cup C|$ is equal to $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.
  TRUE (72% correct). This is the correct three-set Inclusion/Exclusion formula.

- (h) There are exactly six binary strings of length 3 that do not have all three of their bits the same.
  TRUE (48% correct). There are eight three-bit binary strings in all, and we leave out 000 and 111.

- (i) Suppose I flip four fair coins, with each toss being independent. Then the probability that there are two heads and two tails is greater than or equal to 1/2.

  FALSE (64% correct). The number of binary strings with two 0's and two 1's is $C(4,2) = 6$, and there are $2^4 = 16$ total strings, so the probability is $6/16 = 0.375 < 0.5$.

- (j) If X and Y are any two events over the same probability space, then $Prob(X \cup Y) = Prob(X) + Prob(Y)$.

  FALSE (60% correct). This is only true if X and Y are disjoint events, meaning $Prob(X \cap Y) = 0$.

The average score on Question 6 was 10.6/20.