NAME: ______________________________

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INFO 150
A Mathematical Foundation for Informatics
Second Midterm Exam Fall 2024

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 100 total points. Probable scale is somewhere around A=90, C=60, but will be determined after I grade the exam.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/15
3	/20
4	/15
5	/15
6	/20
Total	/100

Question 1 (15): Briefly explain the meaning of each of these terms or concepts (3 points each):

- (a) the **power set** of a set
- (b) a **symmetric** binary relation
- (c) the **Inclusion/Exclusion Principle** for two sets, also called the **Double Counting Rule**
- (d) the **codomain** of a function
- (e) when a binary relation on a set is a **partial order**

Question 2 (15): For any positive integer n, let $S(n)$ be the sum $\sum_{i=1}^{n} i^3$. That is, $S(n)$ is the sum of the numbers $1^3 + 2^3 + \ldots + n^3$. For example, $S(1) = 1$ and $S(3) = 1 + 8 + 27 = 36$. We would like to prove by induction that for all positive naturals n, $S(n)$ is given by the closed formula $S(n) = (\frac{n^2+n}{2})^2$.

- (a, 3) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

- (b, 3) State and prove the **base case** (or base cases) for your induction.

- (c, 3) State the **inductive hypothesis** and **inductive goal** for your inductive step.

- (d, 6) Prove your inductive step, completing the proof.

Question 3 (20): Here we define a sequence of positive integers by the rules $G(1) = 1$, $G(2) = 3$, $G(3) = 9$, and for all n with $n \geq 4$, $G(n) = 8G(n-2) + 3G(n-3)$. Prove by induction, for all positive integers n with $n > 1$, that $G(n) = 3^{n-1}$.

- (a, 4) Write the precise boolean statement $P(n)$ that we would like to prove to be true for all positive integers n.

- (b, 4) State and prove the **base case** (or **base cases**) for your induction.

- (c, 4) State the **inductive hypothesis** and **inductive goal** for your inductive step.

- (d, 8) Prove your inductive step, completing the proof.

Question 4 (15): Let A be the set $\{a, b, c, d\}$, let B be the set $\{1, 2, 3, 4\}$. Let $R \subseteq A \times B$ be the relation $\{(a, 3), (b, 1), (d, 2)\}$, $S \subseteq A \times B$ be the relation $\{(a, 2), (c, 4), (d, 2)\}$, and $T \subseteq A \times B$ be the relation $\{(b, 4), (c, 4), (d, 2)\}$.

- (a, 5) Which of the relations $R \cup S$, $R \cup T$, $S \cup T$, and $R \cup S \cup T$, if any, are functions from A to B? Explain your answers.

- (b, 5) Explain why $R \cup (S \cap T)$ is a function from A to B.

- (c, 5) Is $R \cup (S \cap T)$ an invertible function? If so, describe its inverse and explain why it is the inverse. If not, justify your claim that it is not.

Question 5 (15): Let $f : A \to B$ and $g : B \to C$ be any functions, and let $h : A \to C$ be the **composition** of f and g, which is the function whose rule is "$h(x) = g(f(x))$".

Recall the definition of an **onto function**: $f : A \to B$ is one-to-one if $\forall y \in B, \exists x \in A, f(x) = y$.

1. (a, 10) Explain why, if f and g are both onto functions, h is also a onto function.

2. (b, 5) If h is a onto function, is it necessarily true that f and g are both onto functions? Explain your answer.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Define a sequence of real numbers by the rule $a_1 = 2$, $a_2 = 1$ and, for all n with $n > 2$, $a_n = (a_{n-1} + a_{n-2})/2$. Then for all positive integers n, a_n is given by the closed form $a_n = (4/3) + (2/3)(-1/2)^n$.

- (b) If the preconditions of a program are true, and it terminates, then its postconditions will be true.

- (c) Let $P(n)$ be a predicate over the positive integers. If we prove $(P(m-2) \wedge P(m-3)) \rightarrow P(m)$ for all m with $m \leq 4$, and we prove $P(1)$, $P(2)$, and $P(3)$, then it may still be possible that for some positive integer n, $P(n)$ is false.

- (d) Let $Q(n)$ be a predicate over the positive integers, such that $Q(3)$ is false, and such that for all n such that $n \geq 9$, $Q(n) \rightarrow Q(n-8)$ and $Q(n-8) \rightarrow Q(n)$ are both true. Then it is not possible that for some positive integer k, $Q(k)$, $Q(k+1)$, $Q(k+2)$, $Q(k+2)$, $Q(k+3)$, $Q(k+4)$, $Q(k+5)$, and $Q(k+6)$ are all true.

- (e) The statement $A \cup (B \cap C') = (A \cap B) \cup (A \cap C')$ is a set identity.

- (f) If A and $A \cup B$ are both infinite sets, then it is possible that B is finite, and it is also possible that B is infinite.

- (g) If $R \subseteq A \times A$ and $S \subseteq A \times A$ are two symmetric relations on the same set, then $R \cap S$ is a symmetric relation but $R \cup S$ need not be symmetric.

- (h) If a and b are any two real numbers with $a \neq 0$, then the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with the rule $f(x) = ax + b$ is invertible.

- (i) Let $f : A \rightarrow B$ be a function where A is an infinite set and B is a finite set. Then f must be an onto function. (**Note:** The term "onto function" is defined in Question 5.)

- (j) A partial order R must contain at least two elements x and y such that $R(x, y)$ and $R(y, x)$ are both false.