NAME: ________________________

SPIRE ID: ________________________

INFO 150
A Mathematical Foundation for Informatics
Solutions to First Midterm Exam Fall 2024

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DIRECTIONS:

- Answer the problems on the exam pages.
- There are six problems on pages 2-7, some with multiple parts, for 100 total points. Probable scale is somewhere around A=90, C=60, but will be determined after I grade the exam.
- If you need extra space use the back of a page.
- No books, notes, calculators, or collaboration.
- In case of a numerical answer, an arithmetic expression like "$2^{17} - 4$" need not be reduced to a single integer.

1	/15
2	/20
3	/15
4	/15
5	/15
6	/20
Total	/100

Question 1 (15): Briefly explain the meaning of each of these terms or concepts (3 points each):

- (a) a **closed formula** for a number sequence

  A closed formula for a number sequence a is a rule giving the value of each a_n as a function of n.

- (b) the **negation** of a proposition

  The negation of a proposition p is a new proposition that is true if and only if p is false, and is false if and only if p is true.

- (c) a **universally quantified predicate**

  A universally quantified predicate is a proposition made from a unary predicate. If $P(x)$ is the predicate, with x of type D as its only free variable, then "$\forall x \in D, P(x)$" is a universally quantified predicate. It is a proposition that is true if $P(x)$ for all values x in D, and is false otherwise.

- (d) the **converse** of an implication

  If $p \rightarrow q$ is an implication, where p and q are any propositions, then the converse of $p \rightarrow q$ is another implication, $q \rightarrow p$. It is not in general equivalent to the original implication.

- (e) the **Division Theorem** in number theory

  The Division Theorem says that if a is any integer, and b is a positive integer, there exists integers q and r such that $a = bq + r$ and $0 \leq r < b$.

Question 2 (20): Translate these four statements as indicated. We have a set of dogs D including (among others) the named dogs Blaze, Gwen, Rhonda, and Wallace who are denoted symbolically by b, g, r and w respectively. We define three predicates on dogs so that $H(x)$ means "dog x likes to dig holes", $S(x,y)$ means "dog x is smaller than dog y", and $L(x,y)$ means "dog x is larger than dog y". (5 points each)

- (a) (to symbols) There is a dog who is both larger than Wallace and smaller than Rhonda, and also does not like to dig holes.

  Solution:

  $\exists x \in D, L(x,w) \wedge S(x,r) \wedge H(x)$.

- (b) (to English) $\forall x \in D, \forall y \in D, (S(x,y) \rightarrow L(y,x)) \wedge (L(x,y) \rightarrow S(y,x))$

  Solution:

  Given any two dogs, if the first is smaller than the second then the second is larger than the first, and if the first is larger than the second, then the second is smaller than the first.

- (c) (to symbols) If Rhonda likes to dig holes, then she is larger than Wallace.

  Solution:

  $H(r) \rightarrow L(r,w)$

- (d) (to English) $\neg H(g) \wedge (S(b,r) \vee H(w))$

  Solution:

  If Gwen does not like to dig holes, then either Blaze is smaller than Rhonda, or Wallace likes to dig holes (or both).

Question 3 (15): Prove the following using truth tables:

- The two compound propositions $p \wedge (\neg q \to \neg p)$ and $\neg(q \to \neg p)$ are logically equivalent.

  First Solution (using EC's notation):

p	q	$\neg q$	$\neg p$	$\neg q \to \neg p$	$p \wedge (\neg q \to \neg p)$	$q \to \neg p$	$\neg(q \to \neg p)$
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	F
F	T	F	T	T	F	T	F
F	F	T	T	T	F	T	F

  Because the sixth and eighth columns are identical, the corresponding compound propositions are logically equivalent.

  Second Solution (using the other notation with columns under each symbol):

p	$\wedge$	$(\neg p$	$\to$	$\neg p)$	$\neg$	$(q$	$\to$	$\neg p)$
T	T	F	T	F	T	T	F	F
T	F	T	F	F	F	F	T	F
F	F	F	T	T	F	T	T	T
F	F	T	T	T	F	F	T	T

  Because the second and the sixth columns are identical, the corresponding compound propositions are logically equivalent.

Question 4 (15): Consider the following statement:

"$\forall n \in \mathbb{Z}$, the number $(n+3)(n+4)$ is even".

1. What would be a **counterexample** disproving this statement?

   Solution: A counterexample would be an integer n such that $(n+3)(n+4)$ was not even.

2. Find three numbers that are *not* counterexamples, showing why they are not counterexamples.

   Solution: Of course *any* integers will turn out to be counterexamples. Three examples are 0 (because $(0+3)(0+4) = 12$ which is even, 1 (because $(1+3)(1+4) = 20$ which is even, and -1 (because $(-1+3)(-1+4) = 6$) which is even.

3. Write a letter from the Author to the Reader, that should convince the Reader to stop looking for any counterexamples.

   Solution:

   Dear Reader,

   You've tried looking for a number n such that $(n+3)(n+4)$ is odd, and you haven't found one yet because there isn't any. We know from the Division Theorem that any number n is either even or odd. If n is even, $n+3$ is odd and $n+4$ is even. If n is odd, $n+3$ is even and $n+4$ is odd. So in both cases, the number you are looking for is the product of an even number and an odd number. But we know that any even number, multiplied by any number at all, is even, making the new number even. (If you've forgotten the proof of that, let n be an even number and m be any integer at all. Then $n = 2k$ for some integer k, because n is even. But then nm is $2km$, and is even because it is 2 times an integer.)

   So please stop looking for counterexamples!

   Regards,

   The Author

Question 5 (15): Prove the following statement:

"If x is any nonzero rational number, then the number $x + \frac{1}{x}$ is also a rational number."

(**Hint:** Recall the definition of a **rational number**: It is a real number that can be expressed as p/q where both p and q are integers, and $q \neq 0$.

Solution:

Since x is rational, we can write $x = p/q$ where p is an integer and q is a nonzero integer. We also know that p is nonzero, because if it were x would be 0, and we are told that it is not.

Then $x + \frac{1}{x}$ can be written as $\frac{p}{q} + \frac{q}{p}$, which we can write as $\frac{p^2+q^2}{pq}$. We know that both $p^2 + q^2$ and pq are integers, by the Closure Property of the Integers. We also know that pq is non zero, because it is the product of two nonzero integers. So $x + \frac{1}{x}$ satisfies the definition to be a rational number.

Question 6 (20): Here are ten **true/false** questions, worth two points each. There is no credit for blank answers, so you should answer all the questions.

- (a) Let a be a number sequence with a recursive formula such that $a_1 = 3$ and, for larger n, $a_n = a_{n-1} + 2$. Then a has a closed formula with $a_n = 2n + 1$.

  TRUE. The closed formula tells us that $a_1 = 2(1) + 1 =$, and that $a_n - a_{n-1} = 2n + 1 - (2(n-1) + 1) = 2$.

- (b) Let b be a number sequence with the closed formula $b_n = 2^n - 3$. Then b has a recursive formula with $b_1 = 2$ and, for larger n, $b_n = b_{n-1} + 2^{n-1}$.

  FALSE. The closed formula tells us that $b_1 = 2^1 - 3 = -1$, but the alleged recursive formula has $b_1 = 2$.

- (c) The negation of the statement "Both Blaze and Rhonda have curly tails" is "Either Blaze does not have a curly tail, or Rhonda does not have a curly tail, or both".

  TRUE. By DeMorgan, the negation of $b \wedge r$ is $\neg b \vee \neg r$.

- (d) If p, q, and r are propositions, and the statement $(p \vee q) \wedge (q \vee r)$ is true, then the three propositions cannot all be true.

  FALSE. If all three variables are true, $p \vee q$ and $q \vee r$ are both true, so the given statement is also true.

- (e) Let D be a set of dogs, and let B be a set of breeds including "pointer (p)" and "spaniel (s)". Define a predicate I such that $I(x, y)$ means "dog x is of breed y". Then the English statement "Some dog is both a pointer and a spaniel" is logically equivalent to the symbolic statement "$(\exists x \in D, I(x, p)) \wedge (\exists y \in D, I(y, s))$".

  FALSE. The symbolic statement says that some dog is a pointer, and some dog is a spaniel, but says nothing about those two dogs being the same.

- (f) In the setting of part (e) of this problem, let Wallace be a particular member of D. That a symbolic statement equivalent to "Every dog except Wallace is a pointer or a spaniel" would have a free variable.

  FALSE. The quantified statement would be "$\forall x \in D, (x \neq w) \rightarrow (I(x,p) \vee I(x,s))$.", which has no free variables.

- (g) The statement "Every dog who is a terrier is also cute" is equivalent to the statement "There does not exists a dog who is not cute and is a terrier".

  TRUE. By the Quantifier DeMorgan rule, using Commutativity of $\wedge$.

- (h) The negation of the statement "For every breed, there is a dog of that breed" is "There exists a breed such that there is no dog of that breed".

  TRUE. This is correct by the Quantifier DeMorgan rule.

- (i) Any multiple of any odd integer must be odd.

  FALSE. The number 3 is odd, and 6 is a multiple of 3, but 6 is not odd.

- (j) Let n and b be positive integers. Then there exist positive integers q and r such that $n = qb + r$ and $0 \leq r < b$.

  FALSE. This is very similar to the Division Theorem, which is true. But it says "positive integers" for n, q, and r, where in the Division Theorem n and q are just integers, and r is non-negative but might be 0. If we have $n = 2$ and $b = 3$, for example, there are no *positive* integers such that $n = qb + r$, as q and r would have to be at least 1, making n at least 4. For that matter, if we had $b = 1$, there would be *no* positive integer r with $r < b$.