The following problems are NP Complete:

- SAT
- 3-SAT
- 3-COLOR
- CLIQUE
- Subset Sum
- Knapsack

Knapsack

Given n objects:

object	$o_1 \quad o_2 \quad \cdots \quad o_n$	
weight	$w_1 \quad w_2 \quad \cdots \quad w_n$	≥ 0
value	$v_1 \quad v_2 \quad \cdots \quad v_n$	

W = max weight I can carry in my knapsack.

Optimization Problem:

choose $S \subseteq \{1, \ldots, n\}$

to maximize $\sum_{i \in S} v_i$

such that $\sum_{i \in S} w_i \;\leq\; W$

Decision Problem:

Given $\bar{w}, \bar{v}, W, V$, can I get total value $\geq V$ while total weight is $\leq W$?

Proposition 20.1 *Knapsack is* **NP***-Complete.*

Proof: Let $I = \langle m_1, \ldots m_n, T\rangle$ be an instance of Subset Sum.

Problem: $(\exists ? S \subseteq \{1, \ldots, n\})\big(\sum_{i \in S} m_i \ = \ T\big)$

Let $f(I) = \langle m_1, \ldots m_n, m_1, \ldots, m_n, T, T\rangle$ be an instance of Knapsack.

Claim: $I \in$ Subset Sum $\quad \Leftrightarrow \quad f(I) \in$ Knapsack

$$(\exists S \subseteq \{1, \ldots, n\})\big(\sum_{i \in S} m_i \ = \ T\big)$$

$$\Leftrightarrow$$

$$(\exists S \subseteq \{1, \ldots, n\})\big(\sum_{i \in S} m_i \ \geq \ T \quad \wedge \quad \sum_{i \in S} m_i \ \leq \ T\big) \spadesuit$$

Fact 20.2 *Even though Knapsack is* **NP***-Complete there is an efficient dynamic programming algorithm that can closely approximate the maximum possible V.*

Fact: NP-complete decision problems are all equivalent.

Belief: NP-complete problems require exponential time in the worst case.

Fact: Difficulty of NP Approximation problems varies widely.

Definition 20.3 A is an *NP-optimization problem* iff

- For each instance x, $n = |x|$, $F(x) \subseteq \Sigma^{p(n)}$ is the set of *feasible solutions*. We can test in P whether $s \in F(x)$.
- Each $s \in F(x)$ has a cost $c(s) \in \mathbf{Z}^+$. The cost $c(s)$ is computable in $F(\mathbf{P})$.

For minimization problems,

$$\mathrm{OPT}(x) \quad = \quad \min_{s \in F(x)} c(s)$$

For maximization problems,

$$\mathrm{OPT}(x) \quad = \quad \max_{s \in F(x)} c(s)$$

♠

Definition 20.4 Let M be a polynomial-time algorithm s.t. on any instance x,

$$M(x) \in F(x)$$

M is an ϵ-*approximation algorithm* iff for all x,

$$\frac{|c(M(x)) - \text{OPT}(x)|}{\max(\text{OPT}(x), c(M(x)))} \;\leq\; \epsilon \qquad \spadesuit$$

$0 < \epsilon < 1$

$\epsilon = .01$ is an excellent approximation! Minimization problems: at most $\frac{100}{99}$ times optimal. Maximization problems: at least 99 percent of optimal.

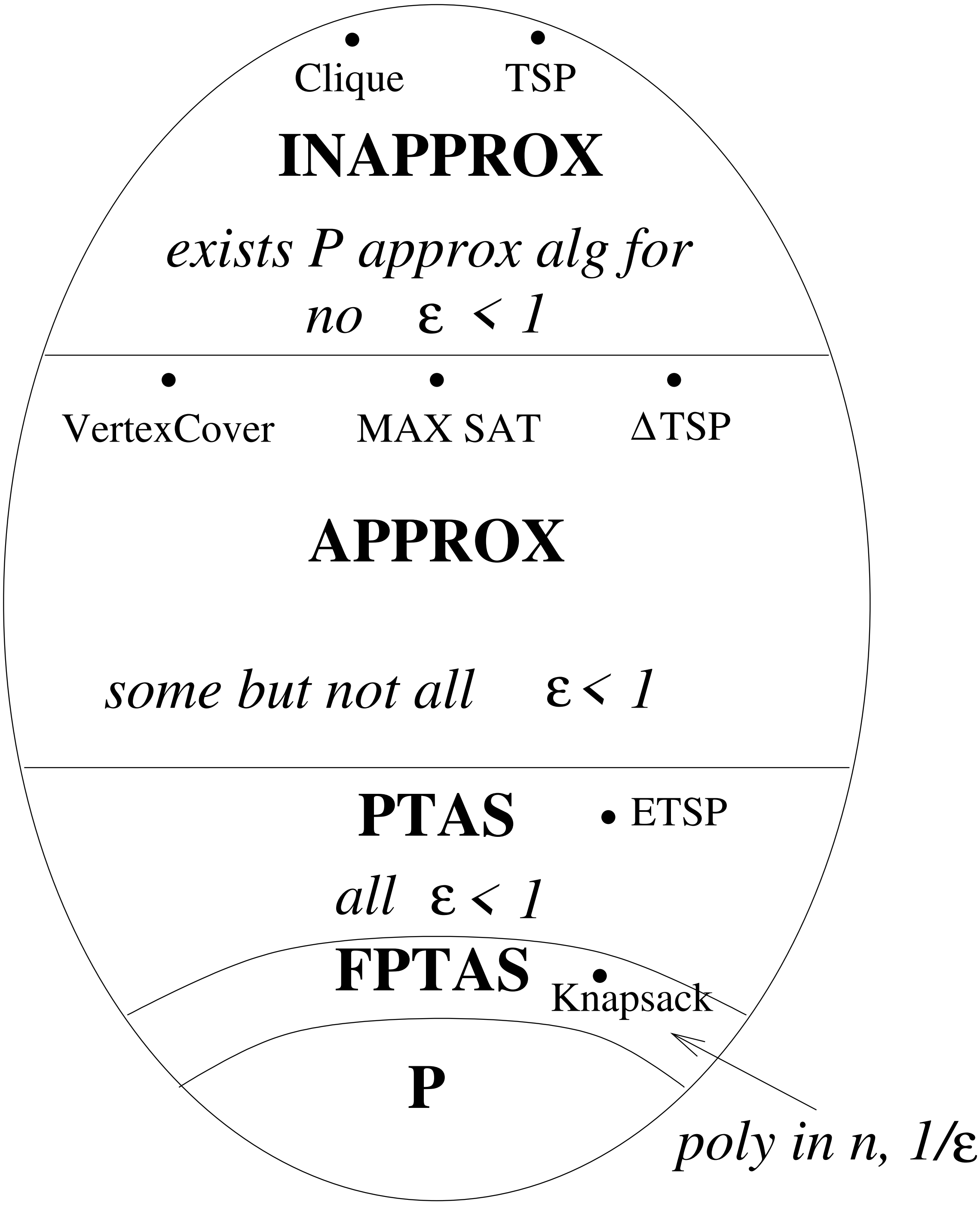
$\epsilon = \frac{1}{2}$: Minimization problems: no more than twice optimal. Maximization problems: at least half optimal.

$\epsilon = .99$: not a very good approximation! Minimization problems: at most 100 times more than optimal; Maximization problems: at least one percent of optimal.

Four Classes of NP Optimization Problems

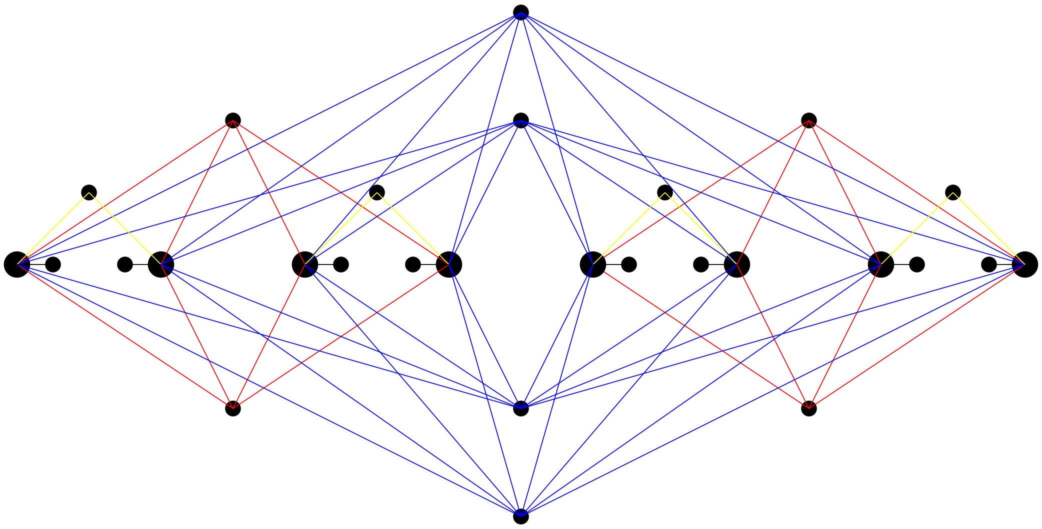
$$\begin{aligned}
\text{INAPPROX} &\equiv \text{no PTIME } \epsilon\text{-approx alg if P} \neq \text{NP} \\
\text{APPROX} &\equiv (\exists \epsilon_1 \epsilon_2 \,.\, 0 < \epsilon_1 < \epsilon_2 < 1) \\
&\ \text{exists PTIME } \epsilon_2\text{-approx alg} \\
&\ \text{no PTIME } \epsilon_1\text{-approx alg if P} \neq \text{NP} \\
\text{PTAS} &\equiv (\forall \epsilon > 0)\text{exists PTIME } \epsilon\text{-approx alg} \\
\text{FPTAS} &\equiv (\forall \epsilon > 0)\text{exists uniform } \epsilon\text{-approx alg} \\
&\qquad \text{running in time poly}(n, \tfrac{1}{\epsilon})
\end{aligned}$$

(F)PTAS stands for *(Fully) Polynomial-Time Approximation Scheme*.


Clique
TSP
INAPPROX
exists P approx alg for
no ε < 1
VertexCover
MAX SAT
ΔTSP
APPROX
some but not all ε < 1
PTAS
ETSP
all ε < 1
FPTAS
Knapsack
P
poly in n, 1/ε

Input: an undirected graph $G = (V, E)$.

Output: min size $C \subset V$ s.t. C touches every edge.



Greedy: (nodes of high degree) About $\log n$ times optimal, in the example above. (There are about $n \log n$ total nodes, the n fat ones are a vertex cover, but the greedy algorithm takes most of the others first.)

Better: Find a Maximal Matching

1. $C := \emptyset$
2. **while** $(E \neq \emptyset)$ **do** $\{$
3. pick $(u, v) \in E$
4. $C := C \cup \{u, v\}$
5. delete u, v from G $\}$

The edges picked are a *maximal matching*, a disjoint set of edges to which we can't add another disjoint edge. If there are m edges in this matching, we've used $2m$ nodes in C but any algorithm would have to use at least m.

$|C| \leq 2\text{opt}(G) \quad \epsilon = \frac{1}{2}$ **Best known approx ratio**

A *Hamilton circuit* for an undirected graph G is a cycle that starts and ends at some vertex v and visits every other vertex exactly once.

$$\text{HC} \quad = \quad \{G \mid G \text{ has a Hamilton Circuit}\}$$

Fact 20.5 *HC is* **NP**-*Complete. (Nicest proof is in Sipser.)*

$$\text{TSP} \quad = \quad \{G = (V, E, w), L \mid G \text{ has a HC of weight } \leq L\}$$

$G = (V, E)$, $n = |V|$, let $h(G) = (V, \binom{n}{2}, w_h), L_h = n$,

$$w_h(u, v) \ = \ \begin{cases} 1 & \text{if } (u, v) \in E \\ n \cdot 10^6 & \text{otherwise} \end{cases}$$

Observation 20.6 *For any undirected graph G,*

$$G \in \ HC \quad \Leftrightarrow \quad h(G) \in \text{TSP}$$

Corollary 20.7 *If* TSP *has a polynomial-time ϵ-approximation algorithm for any $\epsilon < 1$, then P = NP. Thus,* TSP $\in$ INAPPROX.

$G = (V, E)$, $n = |V|$, let $e(G) = (V, \binom{V}{2}, w_e), L_e = n \cdot 10^6$,

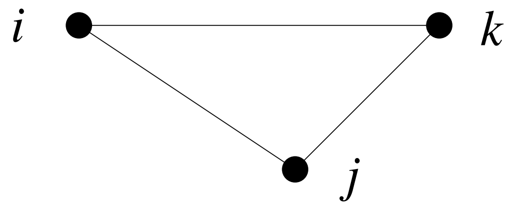
$$w_e(u, v) = \begin{cases} 1,000,000 & \text{if } (u, v) \in E \\ 1,000,001 & \text{otherwise} \end{cases}$$

Let Fair-TSP be the subset of TSP s.t. no edge weight is more than 0.0001 percent more than any other edge weight.

Observation 20.8 *For any undirected graph G,*

$$G \in \ HC \ \Leftrightarrow \ e(G) \in \mathrm{TSP} \ \Leftrightarrow \ e(g) \in \mathrm{Fair{-}TSP}$$

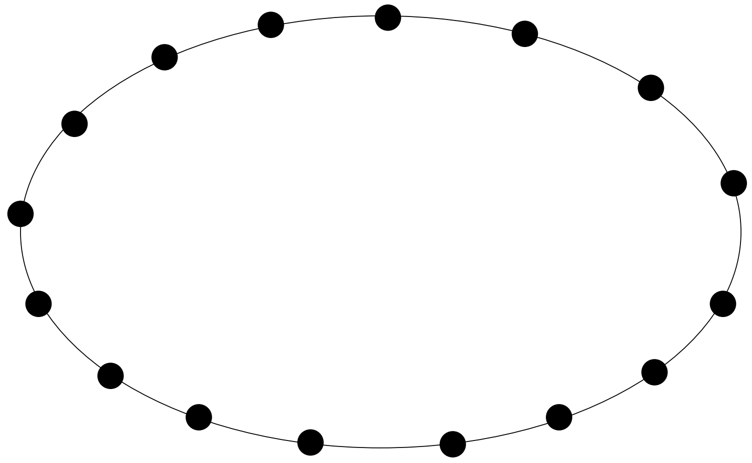
Observation 20.9 Fair-TSP *is NP-complete as a decision problem and as an optimization problem it has a polynomial-time* 10^{-6}*-approximation algorithm.*



ΔTSP: $d(i,k) \leq d(i,j) + d(j,k)$

Claim 20.10 *Minimum Spanning Tree is a lower bound for* ΔTSP : $\quad c(\mathit{MST}) \leq c(\Delta\text{TSP})$.

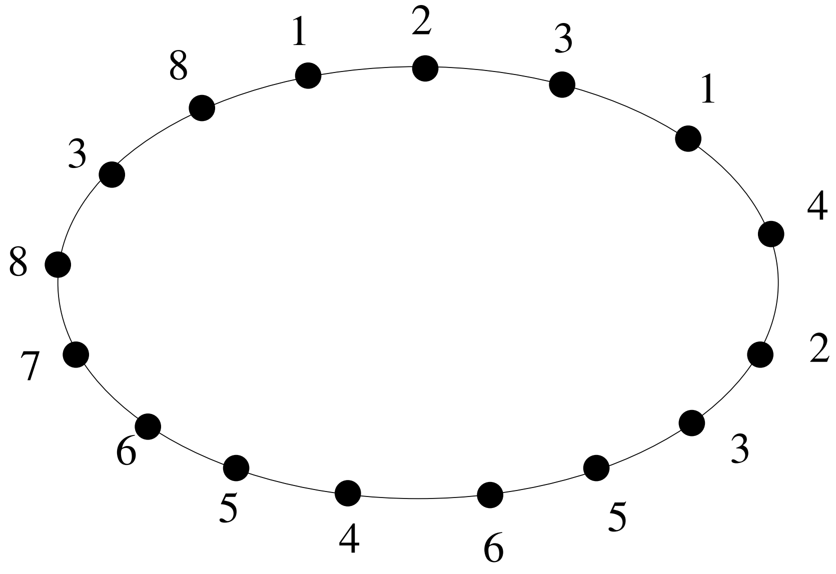
Proof: Visualize optimal tour:



Delete one edge and we have a spanning tree. ♠

Theorem 20.11 $\;$ c(MST) $\leq c(\Delta\mathrm{TSP}) \leq 2\cdot$c(MST)

Proof: The multigraph $2\cdot$MST, made by taking two copies of each edge in the tree, is connected and all its nodes have even degree.



Thus it has an Euler's tour, providing an $\epsilon = \frac{1}{2}$ approximation algorithm. ♠

Aside: A **multigraph** $G = (V, E)$ is a graph except that E may be a multiset, i.e., there can be more than one edge between a certain pair of vertices. An **Euler's Tour** of an undirected multigraph G is a tour that starts and ends at the same vertex and traverses **each edge exactly once**.

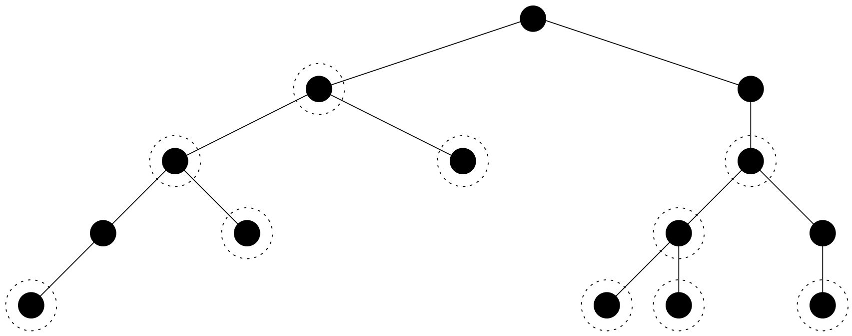
Fact: G has an Euler's tour iff G is connected and each vertex of G has even degree. If G has an Euler's tour, then such a tour can be computed in linear time.

Christofides Algorithm (1976)

In the MST, only worry about the odd degree nodes.

There are an even number of vertices of odd degree.

In polynomial time we can find a minimum weight perfect matching, M, on the odd-degree nodes.



MST $\cup M$ is an Eulerian graph.

MST $\leq$ TSP; $\quad M \leq \frac{1}{2}$TSP.

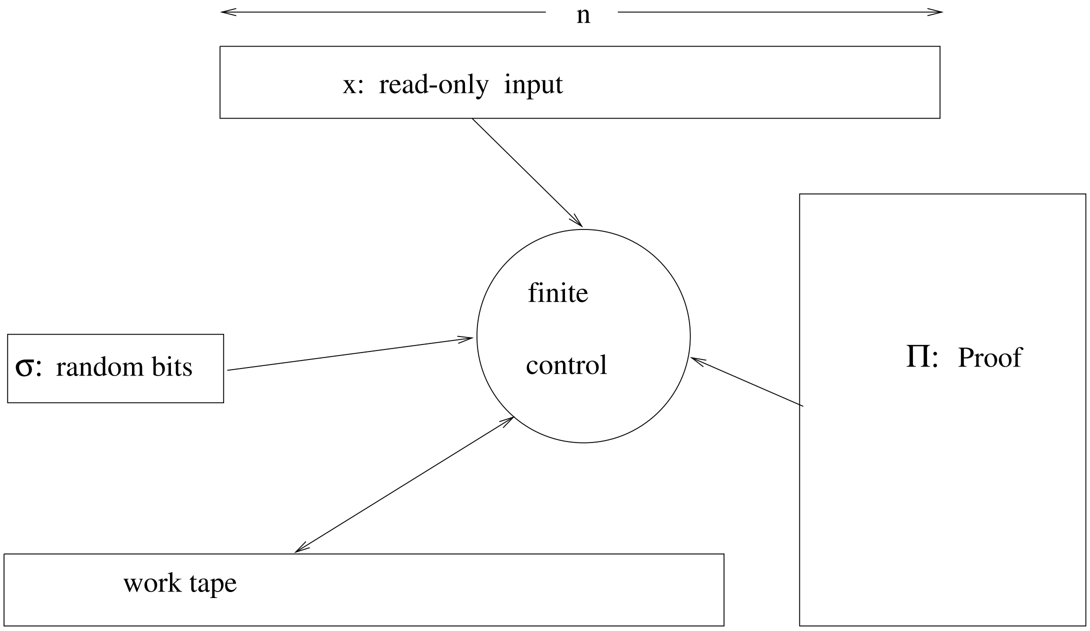
Thus, we get a tour at most 1.5 times optimal. $\epsilon = \frac{1}{3}$

Euclidean TSP

ETSP: Euclidean distance in plane:

$$d((x,y),(x',y')) = \sqrt{(x-x')^2 + (y-y')^2}$$

ETSP has a Polynomial-Time Approximation Scheme (PTAS) [Arora 1997].



Merlin-Arthur games (MA) [Babai]

Decision problem: D; input string: x

Merlin — Prover — chooses the polynomial-length string Π that **Maximizes** the chances of convincing Arthur that x is an element of D.

Arthur — Verifier — "computes" the **Average** value of his possible computations on Π, x. Arthur is a polynomial-time, probabilistic Turing machine.

Definition 20.12 We say that $\mathbf{A}$ *accepts* D iff the following conditions hold:

1. If $x \in D$, there exists a proof Π_x, such that $\mathbf{A}$ accepts for every random string σ,

$$Pr_\sigma \left[\mathbf{A}^{\Pi_x}(x, \sigma) = Accept\right] = 1$$

2. If $x \notin D$, for every proof Π, $\mathbf{A}$ rejects for most of the random strings σ,

$$Pr_\sigma \left[\mathbf{A}^{\Pi}(x, \sigma) = Accept\right] < \frac{1}{2}$$

♠

Any decision problem $D \in$ **NP** has a deterministic, polynomial-time verifier satisfying Definition 20.12.

By adding randomness to the verifier, we can greatly restrict its computational power and the number of bits of Π that it needs to look at, while still enabling it to accept all of **NP**.

We say that a verifier **A** is $(r(n), q(n))$-*restricted* iff for all inputs of size n, and all proofs Π, **A** uses at most $O(r(n))$ random bits and examines at most $O(q(n))$ bits of its proof, Π.

Let PCP$(r(n), q(n))$ be the set of boolean queries that are accepted by $(r(n), q(n))$-restricted verifiers.

Fact 20.13 (PCP Theorem) **NP** $=$ PCP$[\log n, 1]$

The proof of this theorem is pretty messy, certainly more than we can deal with here. But we can look at the applications of the PCP Theorem to approximation problems.

MAX-3-SAT: given a 3CNF formula, find a truth assignment that maximizes the number of true clauses.

$$(x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee x_4 \vee \overline{x_5}) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_4})$$
$$\wedge (\overline{x_2} \vee x_3 \vee x_5) \wedge (\overline{x_3} \vee \overline{x_4} \vee \overline{x_5}) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_2} \vee \overline{x_4} \vee x_5)$$

Proposition 20.14 MAX-3-SAT *has a polynomial-time* $\epsilon = \frac{1}{2}$ *approximation algorithm.*

Proof: Be greedy, set each variable in turn to the better value. ♠

You can do better – a random assignment gets 7/8 of the clauses.

Open for Years: Assuming **NP** $\neq$ **P** is there some ϵ, $0 < \epsilon < 1$ s.t. MAX-3-SAT has no PTIME ϵ-approximation algorithm?

Theorem 20.15 *The PCP theorem (*$\mathbf{NP} = \mathrm{PCP}[\log n, 1]$*)* *is equivalent to the fact that*

If $\mathbf{P} \neq \mathbf{NP}$*, then*

For some ϵ*,* $1 > \epsilon > 0$*,*

MAX-3-SAT *has no polynomial-time,* ϵ*-approximation algorithm.*

Fact 20.16 MAX-3-SAT *has a PTIME approximation algorithm with* $\epsilon = \frac{1}{8}$ *and no better ratio can be achieved unless* $\mathbf{P} = \mathbf{NP}$.

References:

- *Approximation Algorithms for NP Hard Problems*, Dorit Hochbaum, ed., PWS, 1997.
- Juraj Hromkovic, *Algorithmics for Hard Problems*, Springer, 2001.
- Sanjeev Arora, "The Approximability of NP-hard Problems", STOC 98, www.cs.princeton.edu/∼arora.