CMPSCI 575/MATH 513

Combinatorics and Graph Theory

Lecture #4: Graph Planarity
(Tucker Section 1.4)
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Graph Planarity

- Definitions and Motivation
- Coloring Maps and Graphs
- The Circle/Chord Method
- Kuratowski's Theorem: K_5 and $K_{3,3}$
- Euler's Theorem: $r = e - v + 2$
- Planar Graphs have $e \leq 3v - 6$
- Bipartite Planars have $e \leq 2v - 4$

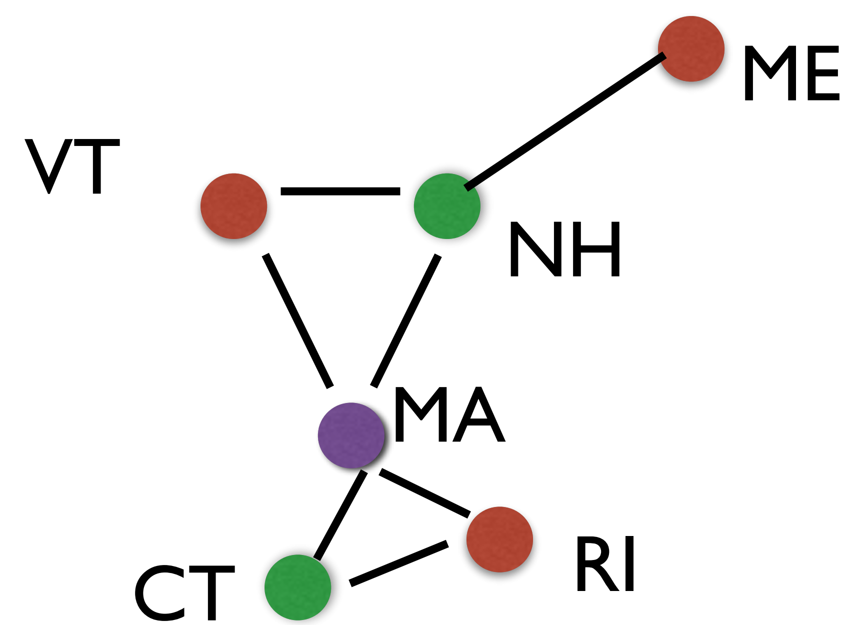
Definitions and Motivation

- A **planar embedding** of a graph is a diagram in the plane where no two edges cross.
- A **planar graph** is a graph for which some planar embedding exists. (A planar graph may have other drawings that do cross edges.)
- An electrical circuit is a graph, and for engineering reasons we don't want wires to cross in our design. Proving things about the class of planar graphs tells us about many naturally occurring graphs.

Coloring Maps and Graphs

- To color a map, we pick a color for each region so that no two adjacent regions have the same color.

- We can convert this to a graph problem by making a node for each region and an edge for each boundary.



Coloring Graphs

- We'll discuss later how 19th century mathematicians conjectured that every planar graph (and hence every map) can be 4-colored.
- We've already seen that a graph can be 2-colored (is bipartite) if and only if it has no odd-length circuit.
- Testing 3-colorability is an NP-complete problem, either for general graphs or for planar graphs.

Testing Planarity

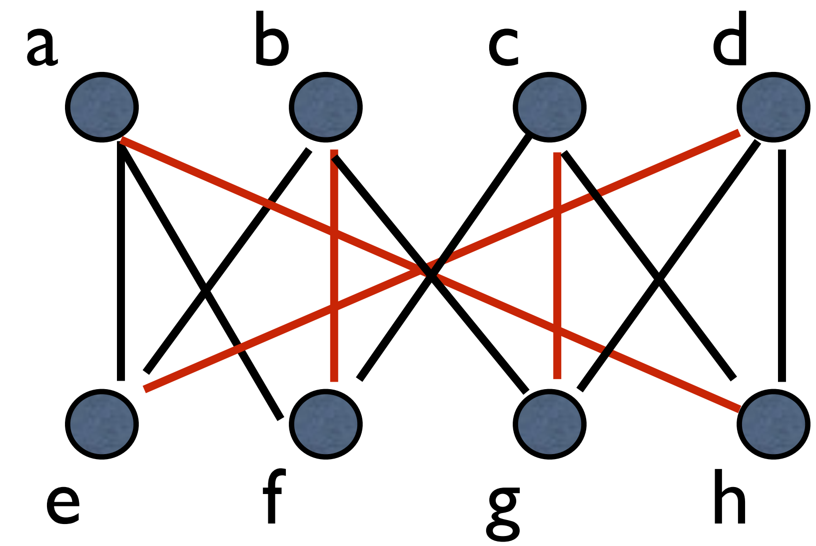
- There are fast algorithms to input a graph, determine whether it is planar, and provide a planar embedding if it is. But they're a bit complicated for this course.
- Here we'll look at a method that suffices in practice to do this for small graphs. We'll also see two famous theorems about planarity, and prove one of them.

The Circle/Chord Method

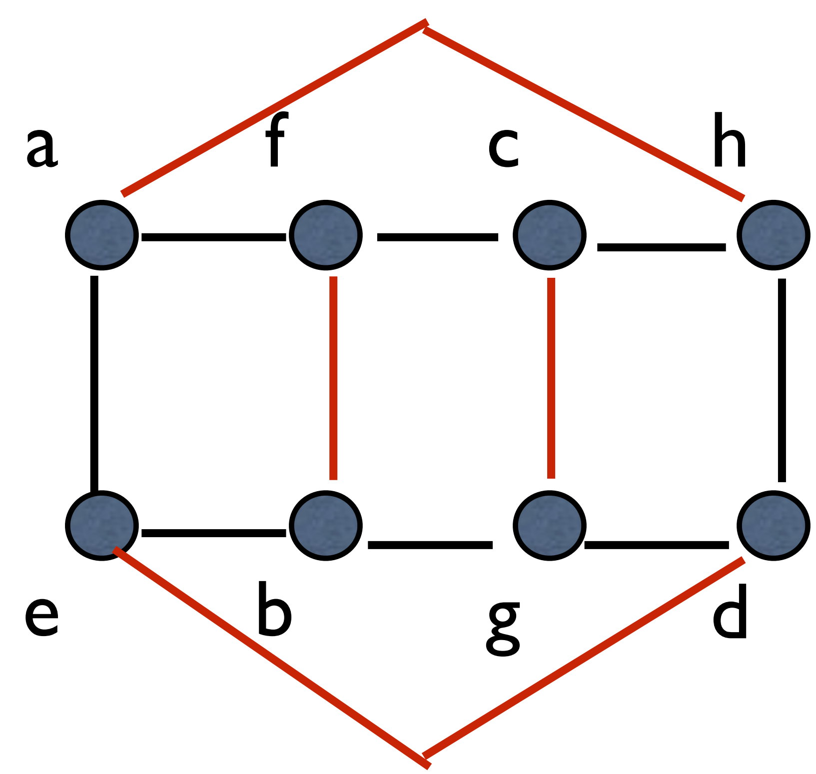
- Many of the graphs we want to consider have a circuit that contains all the vertices, also called a **Hamiltonian circuit**.
- If a graph with such a circuit has a planar embedding, then it must be possible to draw the graph with that circuit as a circle.
- Every other edge of the graph must then be a chord, connecting two vertices on the circle either inside it or outside.

A Circle-Chord Example

- Here is a graph with 8 nodes and 12 edges. It has a Hamilton circuit a-f-c-h-d-g-b-e-a, in black.

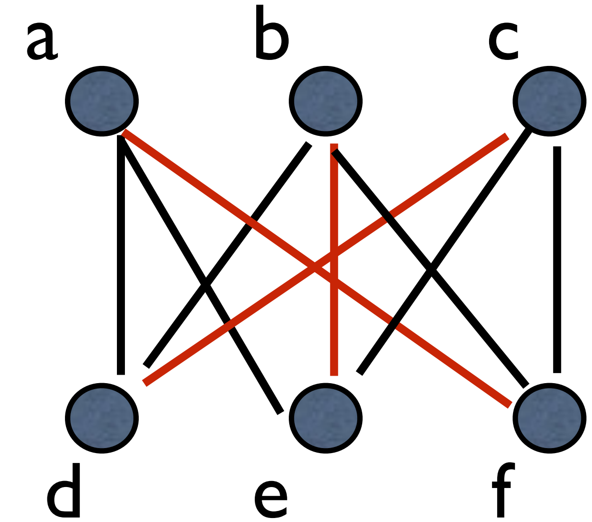


- Redrawing the graph, we now have to place the four red edges. We can put bf and cg in, and ah and ed out.

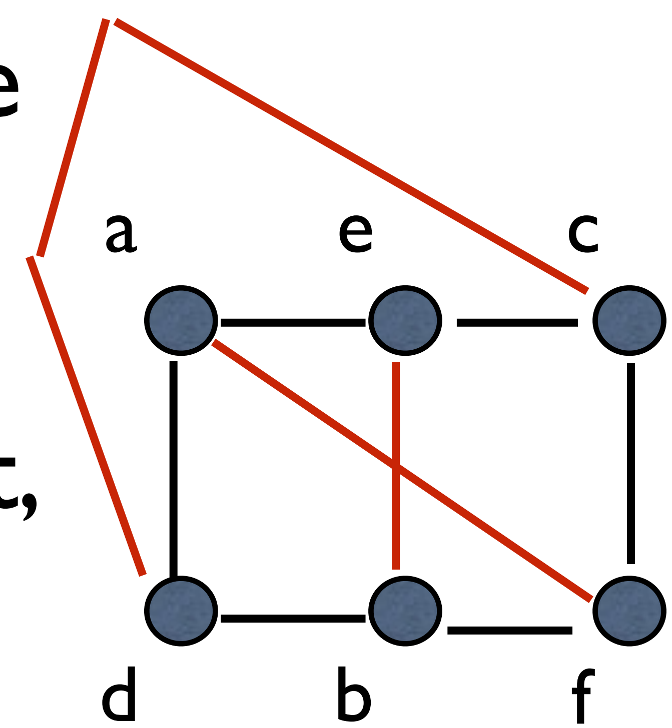


Another Example

- This graph, called $K_{3,3}$, has 6 nodes and 9 edges. Hamilton circuit a-e-c-f-b-d is in black.



- But now we can't place the three red edges without crossing. If af goes in, cd must be out, and there is no place for be either in or out.



Kuratowski's Theorem

- We've just seen that $K_{3,3}$ is a non-planar graph. In the same way we can show that K_5, the complete graph on five nodes, is non-planar.
- Kuratowski proved that *any* non-planar graph "contains" either $K_{3,3}$ or K_5 in a certain way.
- A $K_{3,3}$ configuration is a $K_{3,3}$ where the edges have been subdivided into paths, and similarly for K_5 configurations. "Containing" $K_{3,3}$ means having a $K_{3,3}$ configuration as a subgraph.

Kuratowski's Theorem

- Thus if we can find a $K_{3,3}$ or K_5 configuration in the graph, we know it is non-planar. In practice, most small non-planar graphs contain a $K_{3,3}$ configuration, and the circle-chord method is often able to find it.
- Exhaustively searching for such configurations gives a polynomial-time algorithm to test planarity, though there are better ones.

Euler's Theorem: $r = e - v + 2$

- A cube has 6 faces, 8 vertices, and 12 edges. A dodecahedron has 12 faces, 20 vertices, and 30 edges. A tetrahedron has 4 faces, 4 vertices, and 6 edges. All satisfy the equation $r = e - v + 2$, where r is the number of faces.
- This rule works for any polyhedron (suitably defined) and, as we'll now see, for any planar embedding of a connected graph.
- Lakatos' *Proofs and Refutations* goes into the definition of a polyhedron starting from this theorem, in dialogue form.

Proof of Euler's Theorem

- We'll prove the theorem by induction on the number of edges in the planar graph.
- The base case is e = 0, forcing v = 1 and r = 1, since the whole plane is a region. This works because 1 = 0 - 1 + 2.
- If we add a new edge to a degree-1 node, we add 1 to v and e without changing r.
- If we connect two existing nodes by an edge, we add 1 to r and e without changing v.

$e \leq 3v - 6$ for Planar Graphs

- Define the **degree of a region** to be the number of edges surrounding it, counting an edge twice if the region is on both sides.
- Once we have more than one edge in a connected planar graph, every region (including the one outside the graph) must have degree at least 3. (No loops or parallel edges.)
- So $3r \leq 2e$, and $r \leq 2e/3$ together with $r = e - v + 2$ gives us $2e/3 \geq e - v + 2$, or $e/3 \leq v - 2$, or finally $e \leq 3v - 6$.

$e \leq 3v - 6$

- This tells us right away that K_5 cannot be planar, since there $v = 5$ and $e = 10$, and $5 > 3 \cdot 5 - 6$.
- But it doesn't rule out $K_{3,3}$ being planar, because there $v = 6$ and $e = 9$, and $9 \leq 3 \cdot 6 - 6$ is true.
- It does justify our earlier claim that many natural graphs are sparse, since all planar graphs have $O(n)$ rather than $O(n^2)$ edges.

Bipartite Planar Graphs

- Suppose now that we have a *bipartite* connected planar graph. Assuming we have more than one edge, the minimum degree of a region is now 4, since the boundary of a region is a circuit and must have even length.
- So now $4r \leq 2e$, and $r = e - v + 2$ gives us $e/2 \geq e - v + 2$, $e/2 \leq v - 2$, and $e \leq 2v - 4$.
- And now we see that $K_{3,3}$ cannot be planar because $v = 6$, $e = 9$, and $9 > 2 \cdot 6 = 4$.